Exploring the QCD Phase Diagram for a Signature of the Critical Point

Jian Deng 邓建 (Shandong University)

In collaboration with: Jiunn-Wei Chen, Lance Labun

Phys. Rev. D. 92. 054019 (2015)

+ Hiroaki Hohyama

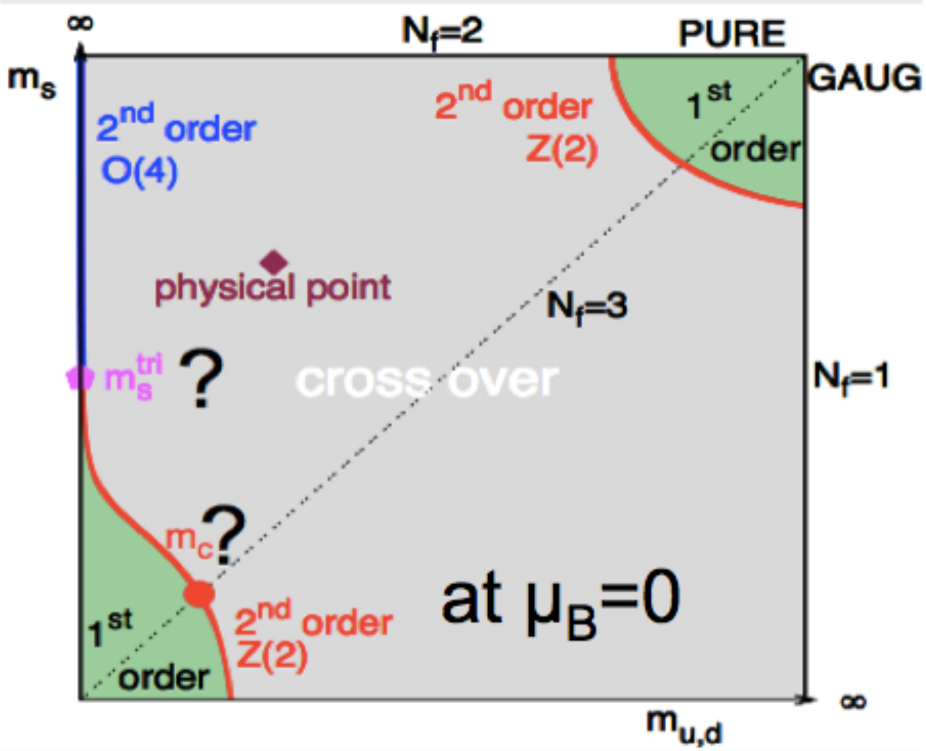
arXiv: 1509.04968

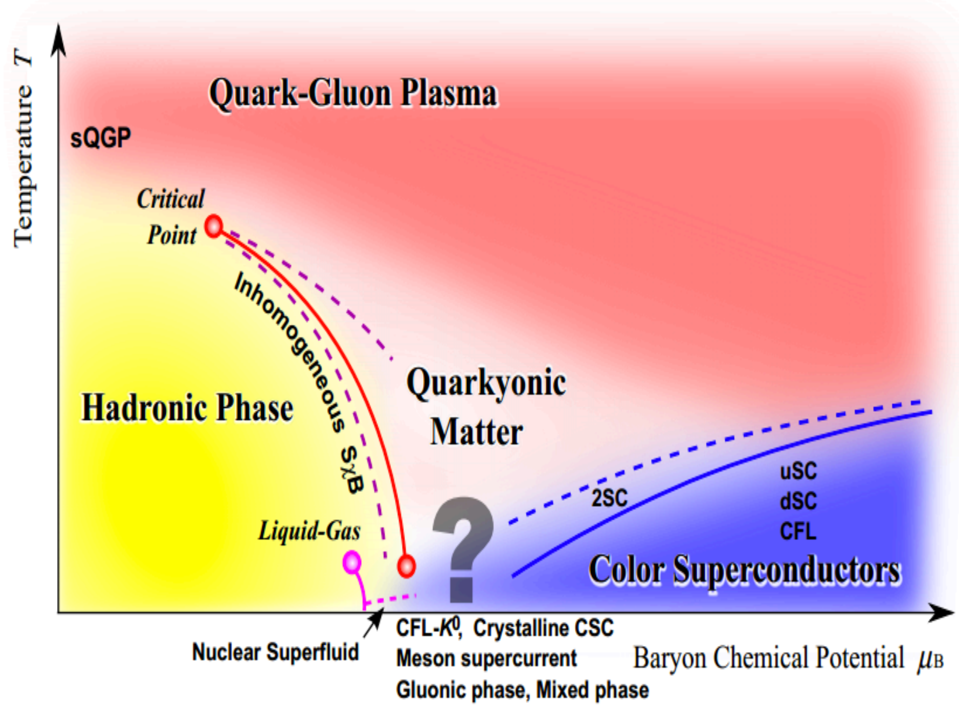
New progress in HIC, 2015, CCNU, Wuhan, 5-9, October,2015

Outlines

1. Introduction and Motivation (phase diagram, critical point, experimental and theoretical approaches)
2. The structure of the phase diagram(from 2nd order phase transition to crossover, from TCP to CEP)
3. A signature of the critical point(model independent analysis and GN model calculation, measurements vs. predictions)
4. 3f-NJL model, more realistic(B,Q,S susceptibilities …)
5. Summary

Phase transition/crossover within QCD





F. R. Brown, et al. PRL 1990

Full lattice QCD simulation

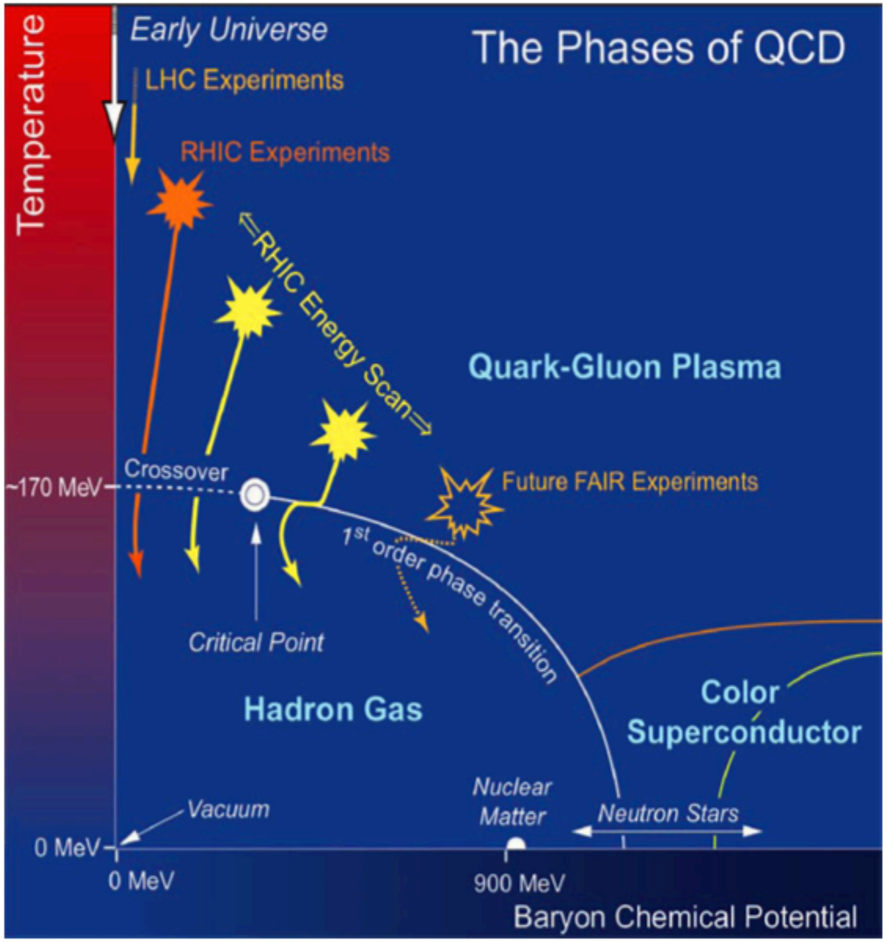
Because the relevant symmetry is explicitly broken by quark mass, symmetry arguments no longer imply the existence of a finite temperature phase transition.

K. Fukushima and T. Hatsuda

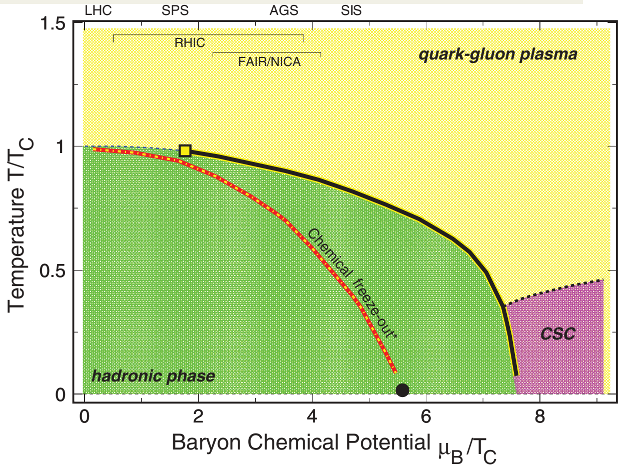
Rep. Prog. Phys. 2010

Even no reliable information from the first-principle LQCD calculation, effective chiral models suggest a first order chiral phase transition in the large density region.

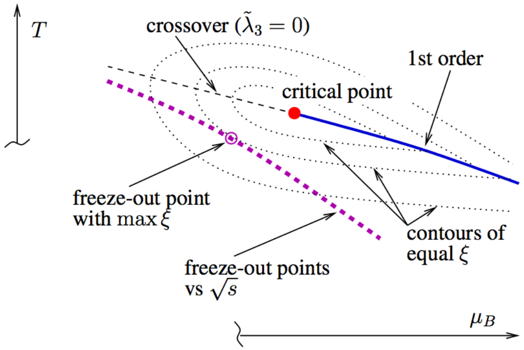
Phase Diagram of QCD



STAR white paper 2014, Studying the phase diagram of QCD matter at RHIC

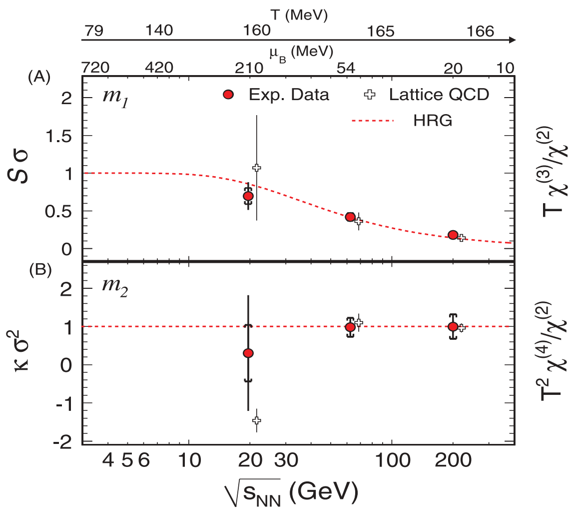


Chemical freeze-out line VS. QCD phase boundary, mapping



measurement VS. thermal equilibrium, singularity

Lattice and Experimental approaches



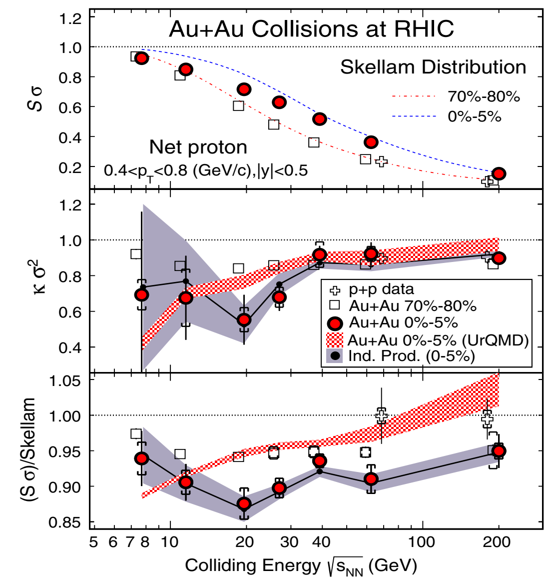
$$T^{n-4}\chi_B^{(n)}\left(\frac{T}{T_c},\frac{\mu_B}{T}\right) = \frac{1}{T^4}\frac{\partial^n}{\partial(\mu_B/T)^n}P\left(\frac{T}{T_c},\frac{\mu_B}{T}\right)\bigg|_{T/T_c}$$

$$(m_1):\ S\sigma = \frac{[B^3]}{[B^2]} = \frac{T\chi_B^{(3)}}{\chi_B^{(2)}},$$

$$(m_2):\ \kappa\sigma^2 = \frac{[B^4]}{[B^2]} = \frac{T^2\chi_B^{(4)}}{\chi_B^{(2)}}$$

$$(m_3):\ \frac{\kappa\sigma}{S} = \frac{[B^4]}{[B^3]} = \frac{T\chi_B^{(4)}}{\chi_B^{(3)}}$$

RHIC BES STAR PRL 2014



Scale for the Phase Diagram of QCD

S. Gupta, X. Luo, B. Mohanty, H. G. Ritter, N. Xu, Science 2011

Sign Problem in Lattice: $\det(D+m+\mu\gamma_0)^* = \det(D+m-\mu^*\gamma_0),$

Extrapolate from mu=0: $P(T,\mu) = P(T) + \frac{\mu^2}{2!}\chi^{(2)}(T) + \frac{\mu^4}{4!}\chi^{(4)}(T) + \cdots$ S. Gupta, Pos Lattice 2010

Shoot from imaginary: $\log Z(\mu_I) = a_0 - a_2\mu_I^2 + a_4\mu_I^4 + O(\mu_I^6).$ M. D'Elia, M-P, Lombardo, PRD 2003

Spin imbalanced Fermi gas on a lattice: Jens Braun, Jiunn-Wei Chen, JD, et al. PRL 2013

Divergence/singularity approaching CEP

Effective. Potential: $\Omega[\sigma] = \int d^3x \left[\frac{(\nabla\sigma)^2}{2} + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 \right]$

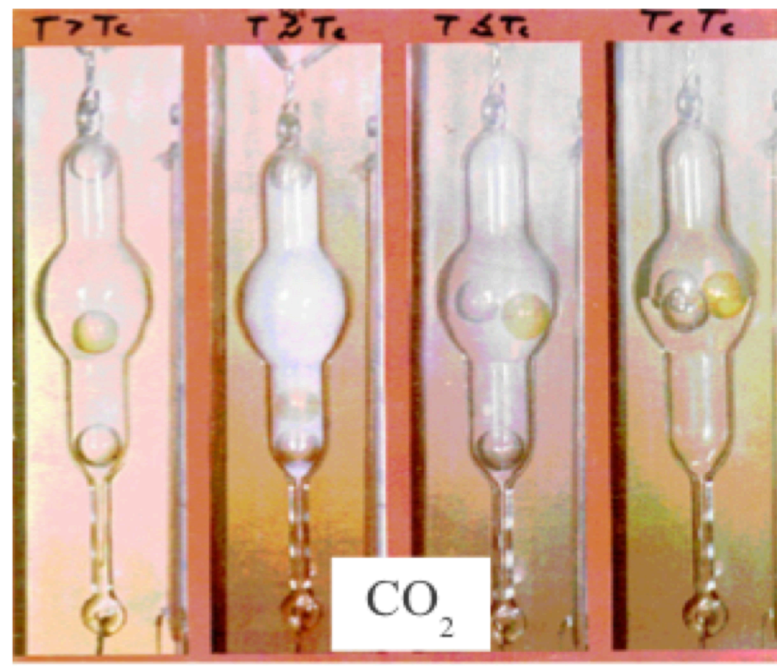
2-point correlator: $\langle \sigma(\mathbf{x})\sigma(0) \rangle \propto \int d^3r \frac{\exp[-\frac{r}{\xi}]}{r}$

Correlation length:

$\xi = m_\sigma^{-1} \to \infty$ @ CEP

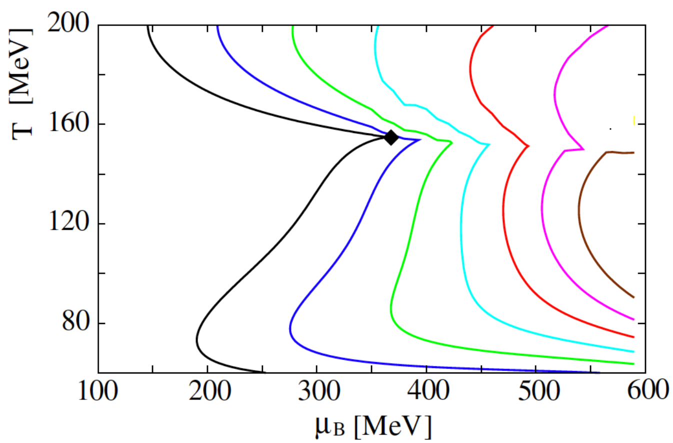
Critical opalescence (临界乳光)

As the CP approached, the density begin to fluctuate over a large length scales, comparable to the wave length of light.

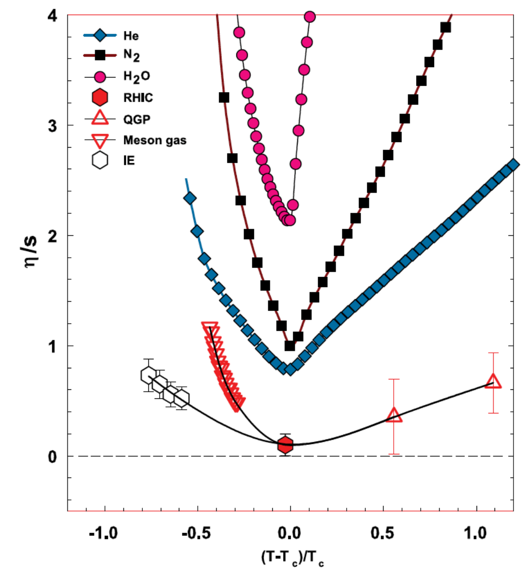


Why study CEP of QCD

Crucial in diagram, test QCD in non-perturbative region

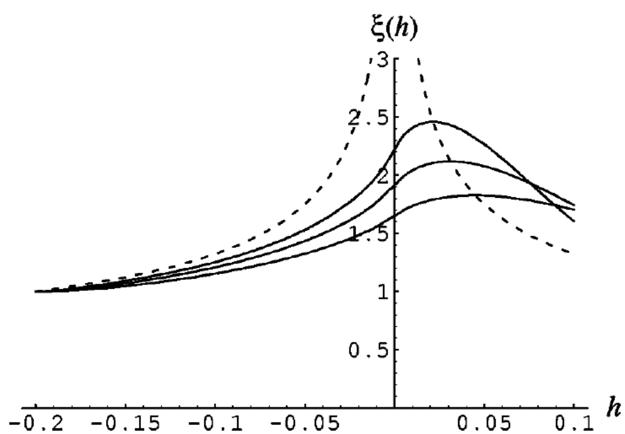


Attractor for thermodynamic trajectories in HIC. C. Nonaka and M. Asakawa, PRC 2005



Most difficult for momentum transport near CEP

Roy A. Lacey, et al. PRL 2007



Slowing out of equilibrium near CEP

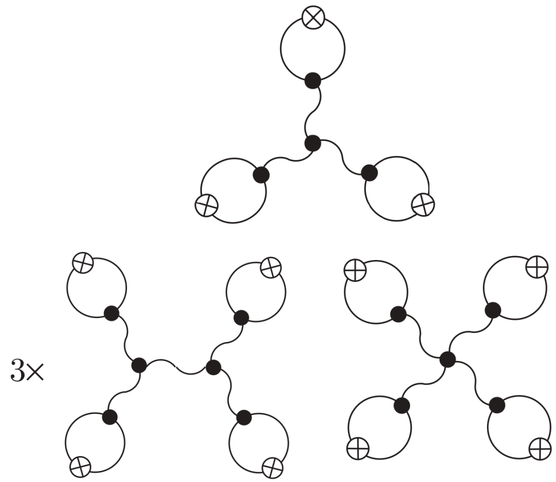
B. Berdnikov and K. Rajagopal, PRD 2000

Higher moments are crucial in HIC

Maximum correlation length 2~3 fm (dynamical evolution, freeze out…)

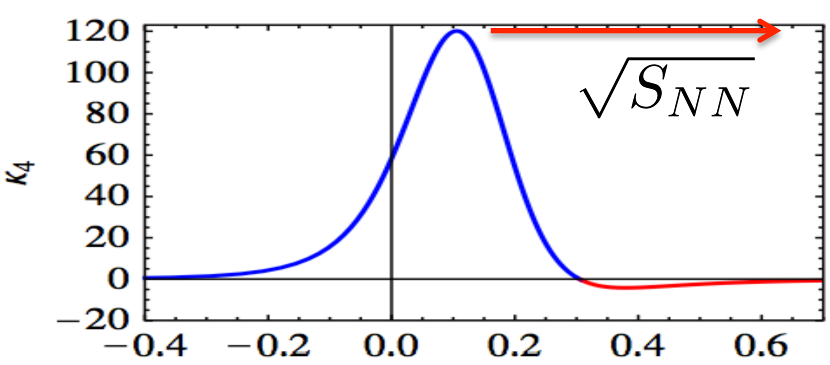
$$\kappa_2 = \langle \sigma_0^2 \rangle = \frac{T}{V}\xi^2; \qquad \kappa_3 = \langle \sigma_0^3 \rangle = \frac{2\lambda_3 T}{V}\xi^6;$$

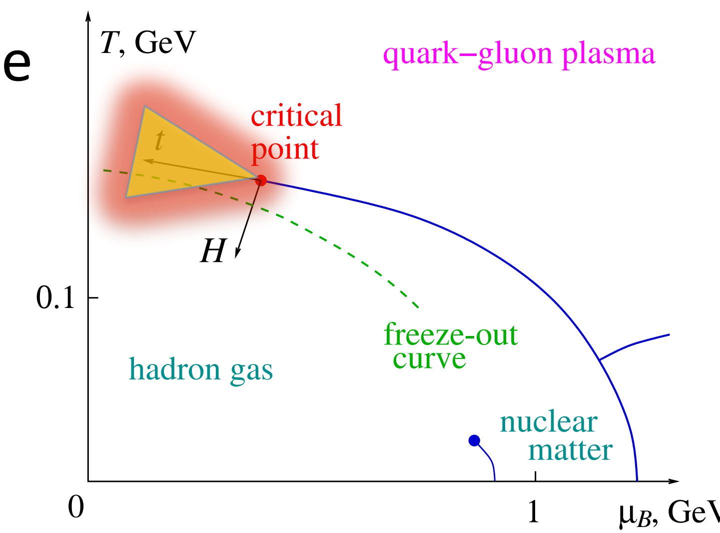
$$\kappa_4 = \langle \sigma_0^4 \rangle_c \equiv \langle \sigma_0^4 \rangle - 3\langle \sigma_0^2 \rangle^2 = \frac{6T}{V}[2(\lambda_3 \xi)^2 - \lambda_4]\xi^8.$$



Non-monotonic functions of the collision energy, higher moments more sensitive signature of CP. M. Stephanov, PRL 2009

Universally, sign change of Kurtosis indicate that CP is close. M. Stephanov, PRL 2011





Phase diagram with tri-critical point

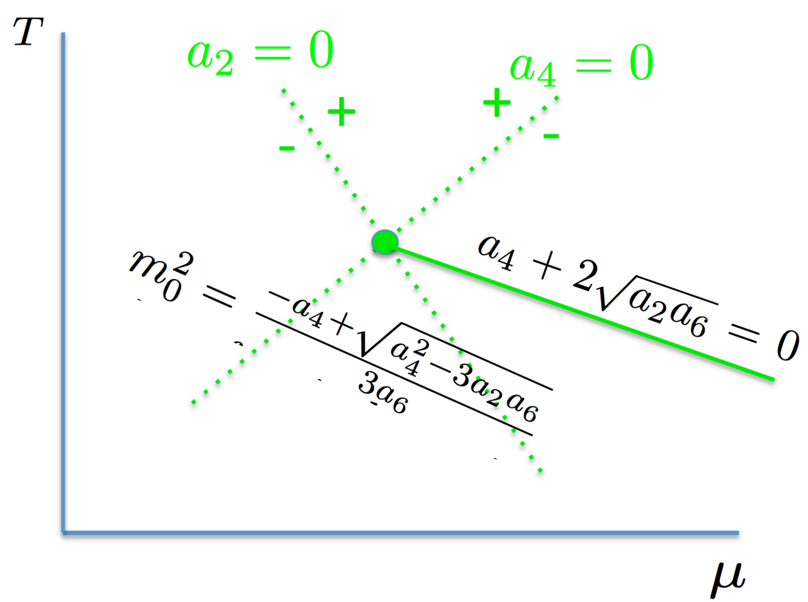
Start with Landau-Ginzburg effective potential, with order parameter m.

long wavelength fluctuations, model-independent.

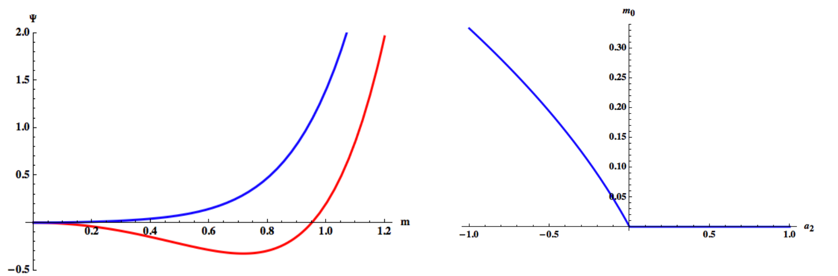
$$\Psi_{2nd} = a_0 + a_2 m^2 + a_4 m^4 + a_6 m^6 + \cdots$$

2[nd] order line follows: $a_2 = 0$

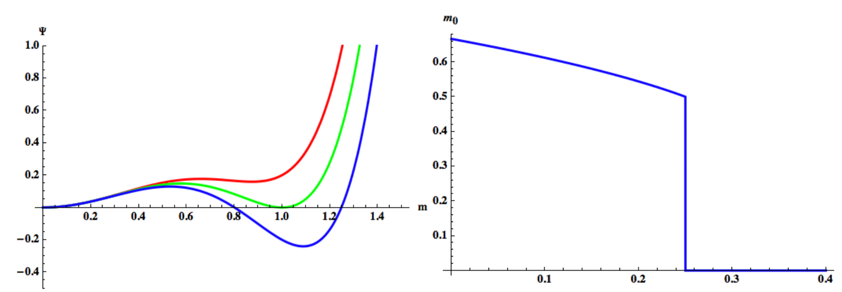
TCP locates at: $a_2 = 0 \quad a_4 = 0$



2[nd] order phase transition



1[st] order phase transition

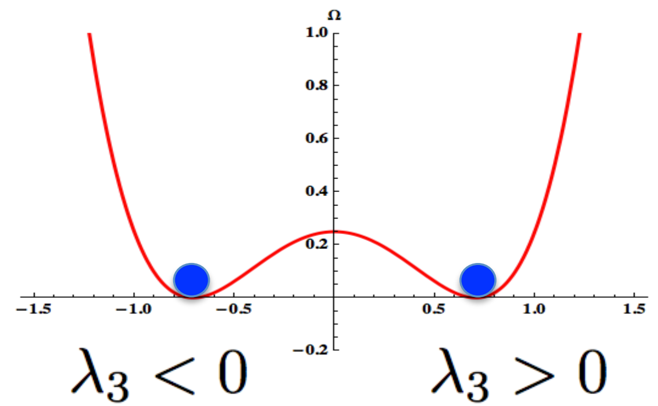


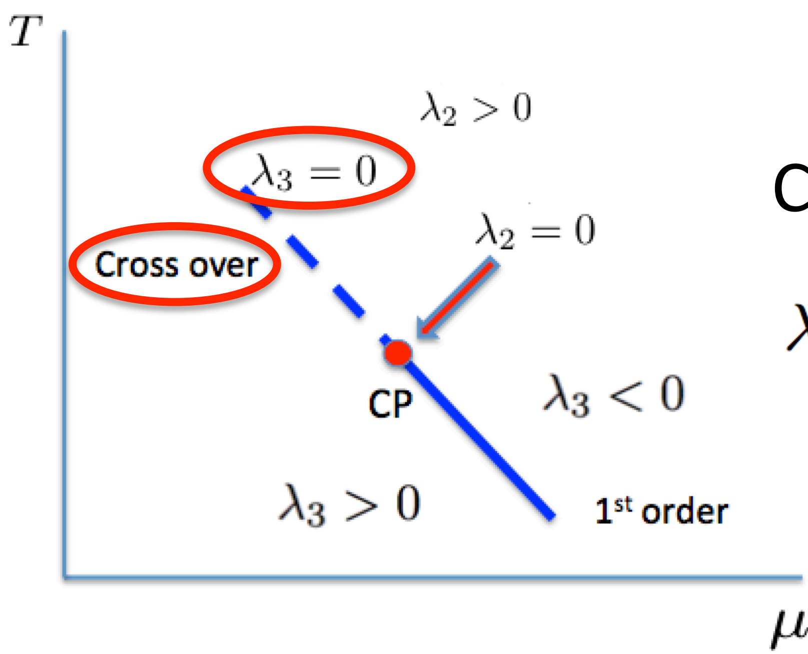
Phase diagram with crossover and CEP

Effective potential for a crossover + critical end point

$$\Omega(x) \;=\; \frac{\lambda_2}{2}x^2 + \frac{\lambda_3}{3}x^3 + \frac{\lambda_4}{4}x^4 .$$

1st order phase transition





CEP locates at: $\lambda_3 = 0, \lambda_2 = 0.$

λ_3 change sign across 1st order line

Crossover line ➔ $\lambda_3 = 0$?

From TCP to CEP

$$\Psi_{2nd} = a_0 + a_2 m^2 + a_4 m^4 + a_6 m^6 + \text{a linear term}$$

$$\Psi_{crossover} = a_0 + a_2 m^2 + a_4 m^4 + m^6 - \frac{\gamma}{\pi} m + \cdots$$

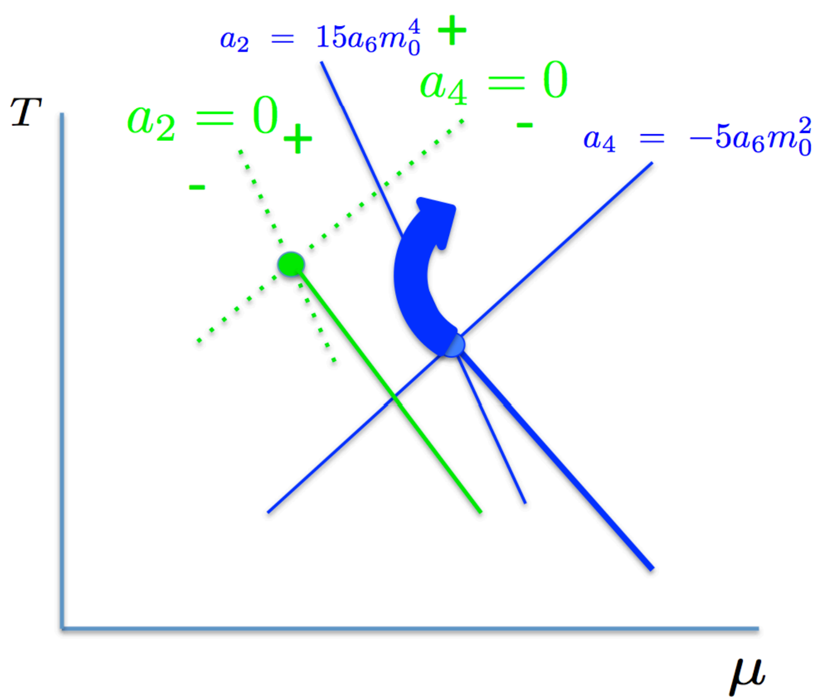
$$\equiv \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \cdots \qquad \sigma = m - m_0$$

CEP locates at: $\lambda_3 = 0, \lambda_2 = 0$

$$a_4 = -5a_6 m_0^2$$
$$a_2 = 15 a_6 m_0^4$$
$$\frac{\gamma}{\pi} = 16 a_6 m_0^5$$

M. Stephanov, K. Rajagopal, E. Shuryak PRD 1999

$\lambda_3 = 0$ follows
$$a_2 = 7a_6 m_0^4 + \frac{\gamma}{\pi}\frac{1}{m_0}$$
$$a_4 = -5a_6 m_0^2$$

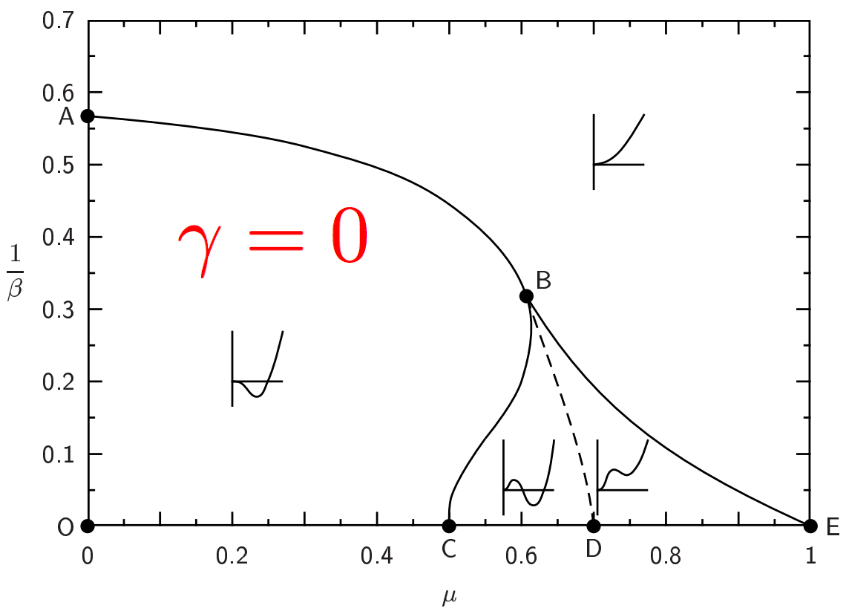


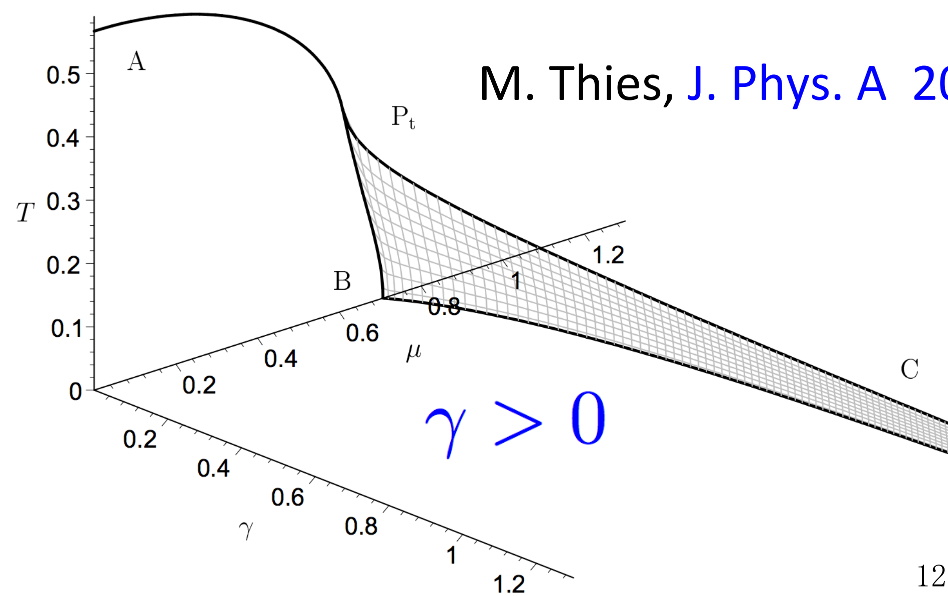
Phase diagram with Gross-Neveu model

GN model, relativistic, renormalizable, QCD like, 1+1D…

Lagrangian: $$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}^{(i)}(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m_0)\psi^{(i)} + \frac{1}{2}g^2\left(\sum_{i=1}^{N}\bar{\psi}^{(i)}\psi^{(i)}\right)^2$$

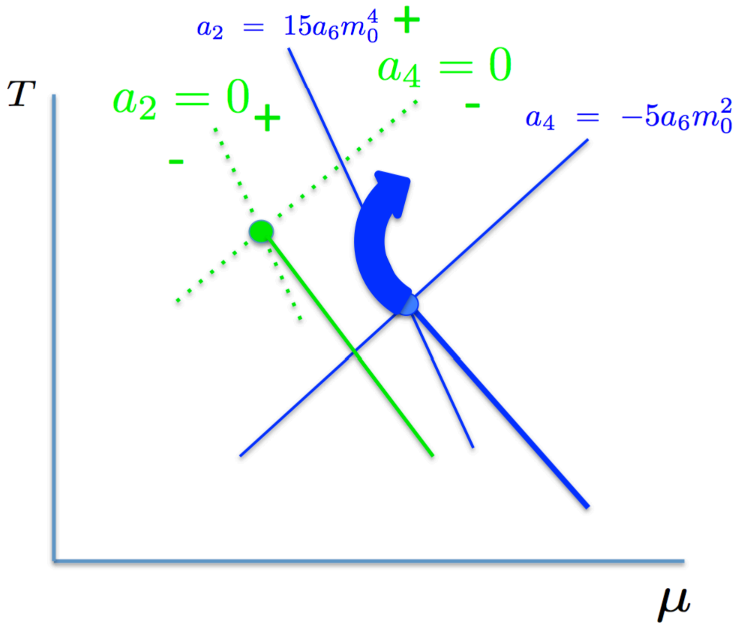
renormalize: $$g,\ m_0 \ \rightarrow\ \gamma := \frac{\pi}{Ng^2}\frac{m_0}{m}, m \equiv 1$$

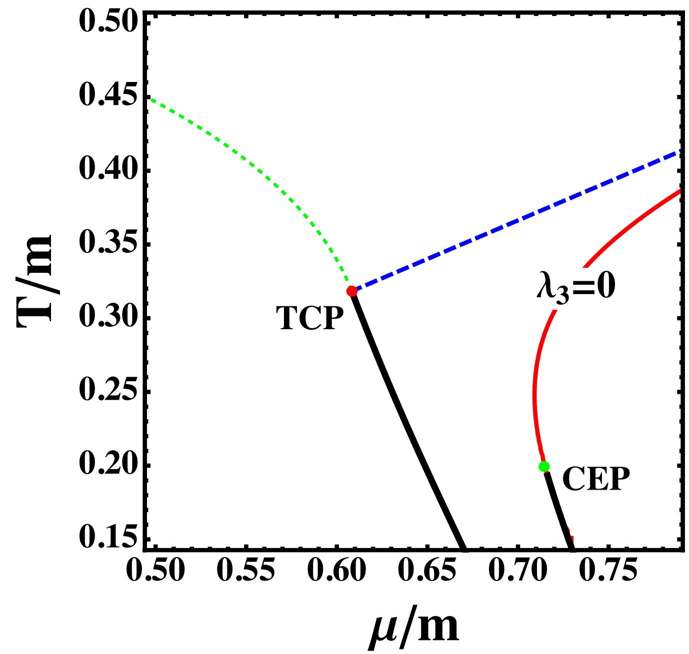




M. Thies, J. Phys. A 2006

Key features with GN model

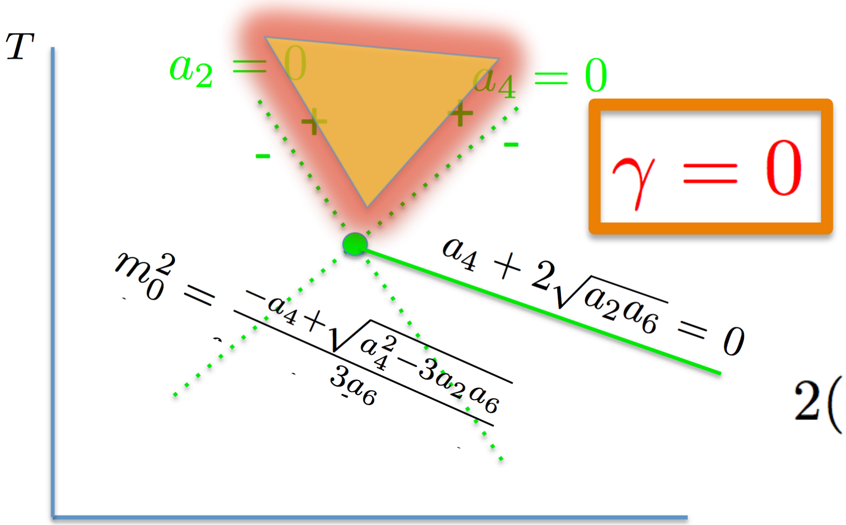




$$\kappa_3 = \langle \sigma_0^3 \rangle = \frac{2\lambda_3 T}{V} \xi^6; \; \kappa_4 = \langle \sigma_V^4 \rangle_c = 6VT^3[2(\lambda_3 \xi)^2 - \lambda_4]\xi^8$$

The line $\lambda_3 = 0$ does NOT lead the crossover line!
but separates the positive and negative region of skewness,
guides the negative region of kurtosis of the sigma field.

Key features with GN model

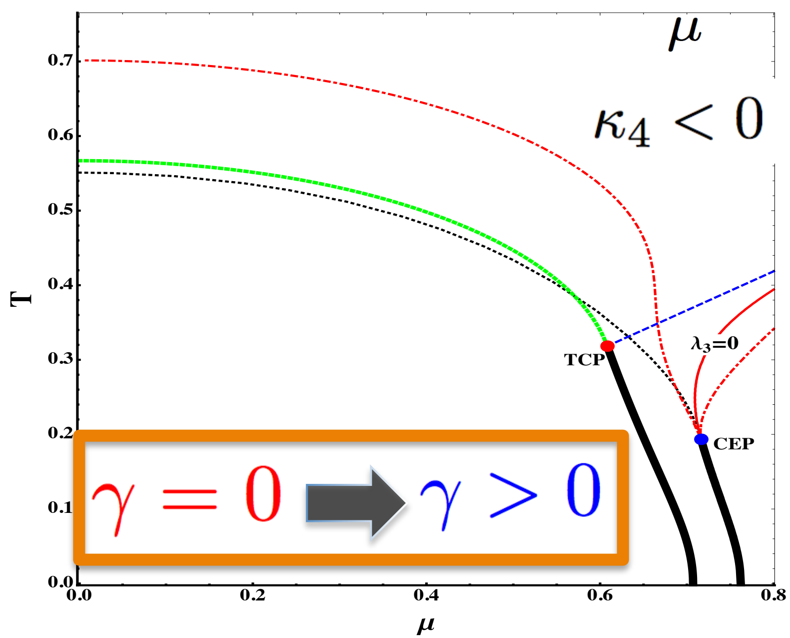


for $m_0 = 0$ case: $\lambda_3 = 0$

$$2(\lambda_3\xi)^2 - \lambda_4 = -a_4$$

for $m_0 > 0$ case:

$$2(\lambda_3\xi)^2 - \lambda_4 = \frac{16}{\sqrt{\Delta}}\left[(a_4 - 2\sqrt{\Delta})^2 + \Delta\right]$$



The negative kurtosis region almost not touch the "hadronic" phase, except in the region very close to the CEP.

It will be very hard for the freeze out curve to enter the negative kurtosis region.

From phase diagram to observables

Susceptibilities = Cumulants

$$T\frac{\partial}{\partial\mu}\log Z = <N>=\bar{N}$$

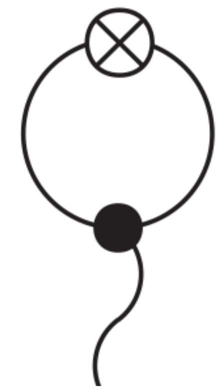
$$T^2\frac{\partial^2}{\partial\mu^2}\log Z = <(N-\bar{N})^2>$$

$$T^3\frac{\partial^3}{\partial\mu^3}\log Z = <(N-\bar{N})^3>$$

$$T^4\frac{\partial^4}{\partial\mu^4}\log Z = <(N-\bar{N})^4> -3<(N-\bar{N})^2>^2$$

Partition function:

$$Z = \mathrm{Tr}\left[\exp\left(-\frac{H-\mu N}{T}\right)\right]$$



By coupling the critical sigma field with the number density, the fluctuation and the criticality will be transferred to the measurements

M. Stephanov, PRL 2011

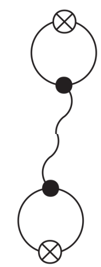
$$\langle(\delta N)^4\rangle_c = \langle N\rangle + \langle\sigma_V^4\rangle_c\left(\frac{gd}{T}\int_p \frac{n_p}{\gamma_p}\right)^4 + \cdots,$$

Most singular part

Susceptibility

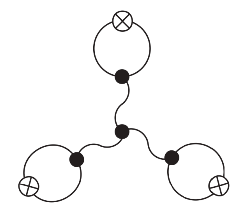
Effective potential: $\Omega(\mu, T, \sigma)$ Grand Canonical ensemble

Partition function: $Z = \int [d\sigma] \exp\left(-\frac{\Omega(T,\mu,\sigma)}{T}\right)$



$$T\frac{\partial}{\partial\mu}\log Z = -\langle\frac{\partial\Omega[\mu,T,\sigma]}{\partial\mu}\rangle = -\langle\Omega'\rangle$$

$$T^2\frac{\partial^2}{\partial\mu^2}\log Z = -T < \Omega'' > + \left(\frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\mu\partial\sigma}\right)^2 \langle\delta\sigma^2\rangle$$

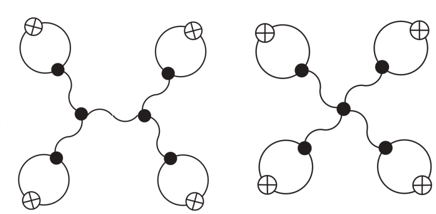


$$T^3\frac{\partial^3}{\partial\mu^3}\log Z = -T^2 < \Omega''' > + \left(\frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\mu\partial\sigma}\right)^3 \langle\delta\sigma^3\rangle + \cdots$$

$$T^4\frac{\partial^4}{\partial\mu^4}\log Z = -T^3 < \Omega'''' > + \left(\frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\mu\partial\sigma}\right)^4 (\langle\delta\sigma^4\rangle - 3\langle\delta\sigma^2\rangle^2) + \cdots$$

How about the prefactors?

3×



Gap equation and singularity

Effective potential $\Omega(\mu, T, \sigma)$

Gap equation: $$\left.\frac{\partial\Omega(\mu,T,\sigma)}{\partial\sigma}\right|_{\sigma=\sigma_0(\mu,T)} = \frac{\partial\Omega}{\partial\sigma}(\mu,T,\sigma_0(\mu,T)) = 0$$

Global minimum, gap equation satisfied for all T and mu

$$\frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\mu\partial\sigma} + \frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\sigma^2}\frac{\partial\sigma_0}{\partial\mu} = 0$$

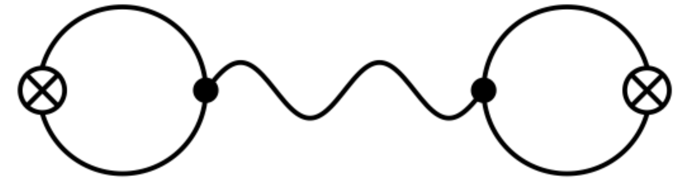
Prefactor: $$\left.\frac{\partial^2\Omega(\mu,T,\sigma)}{\partial\mu\partial\sigma}\right|_{\sigma=\sigma_0(\mu,T)} \propto m_\sigma^2\frac{\partial\sigma_0}{\partial\mu} \propto \xi^{-2}\ \frac{\partial\sigma_0}{\partial\mu}$$

Correlation length dependence canceled in the singular parts.

Singularity from correlation ==> singularity from discontinuity.

Tree level contribution

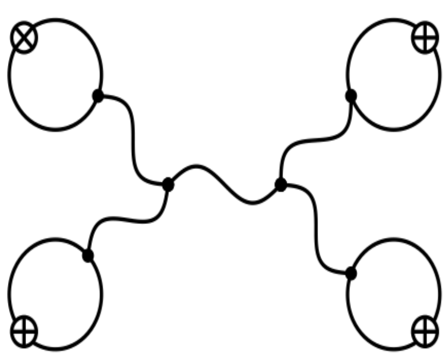
$$T^2 \frac{d^2}{d\mu^2}\log Z = -Ta_2^0 + (a_1^1)^2 < \delta\sigma^2 >$$



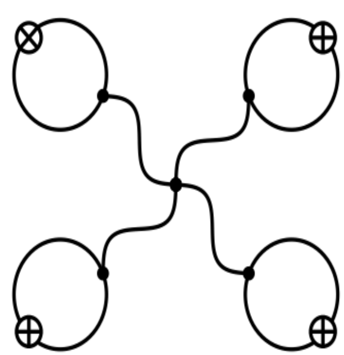
$$T^3 \frac{d^3}{d\mu^3}\log Z = -T^2 a_3^0 + 3T a_1^1 a_2^1 < \delta\sigma^2 > -(a_1^1)^3 < \delta\sigma^3 > -6(a_1^1)^2 a_1^2 < \delta\sigma^2 >^2$$

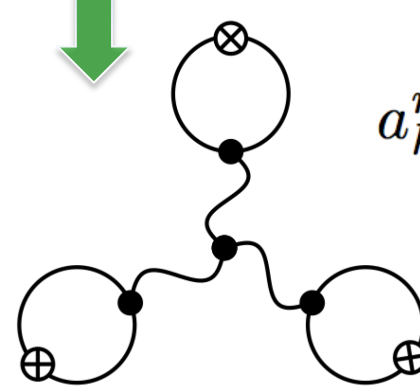
$$\begin{aligned} T^4 \frac{d^4}{d\mu^4}\log Z = & -T^3 a_4^0 + 3T^2 (a_2^1)^2 < \delta\sigma^2 > +4T^2 a_1^1 a_3^1 < \delta\sigma^2 > -6T(a_1^1)^2 a_2^1 < \delta\sigma^3 > \\ & + (a_1^1)^4 (< \delta\sigma^4 > -3 < \delta\sigma^2 >) -12T\left[(a_1^1)^2 a_2^2 + 2a_1^1 a_1^2 a_2^1\right] < \delta\sigma^2 >^2 \\ & + 24\left[(a_1^1)^3 a_1^3 + 2(a_1^1)^2 (a_1^2)^2\right] < \delta\sigma^2 >^3 +24(a_1^1)^3 a_1^2 < \delta\sigma^2 >< \delta\sigma^3 > \end{aligned}$$

3×



+





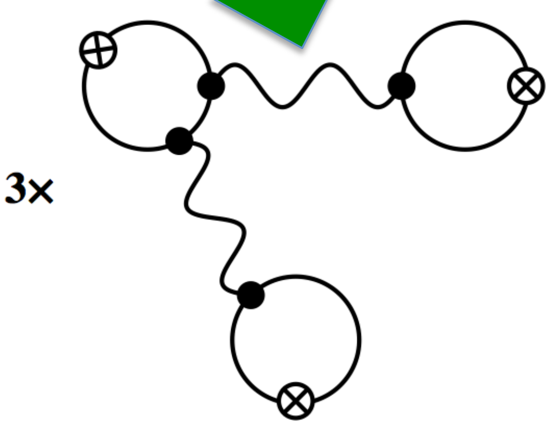
$$a_k^n = \frac{V}{n!}\frac{\partial^{k+n}\Omega}{\partial\mu^k \partial\sigma^n}$$

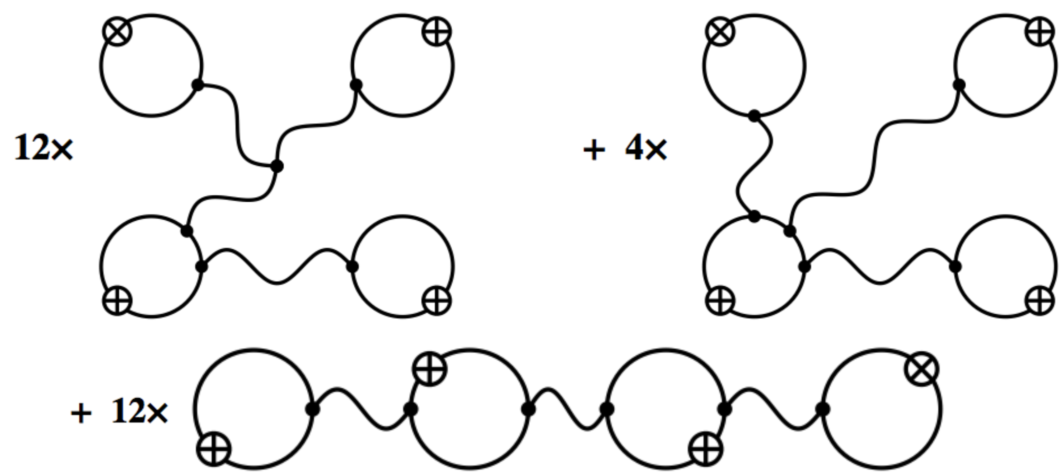
Tree level contribution

$$T^2 \frac{d^2}{d\mu^2} \log Z = -T a_2^0 + (a_1^1)^2 < \delta\sigma^2 >$$

$$T^3 \frac{d^3}{d\mu^3} \log Z = -T^2 a_3^0 + 3T a_1^1 a_2^1 < \delta\sigma^2 > - (a_1^1)^3 < \delta\sigma^3 > - 6 (a_1^1)^2 a_1^2 < \delta\sigma^2 >^2$$

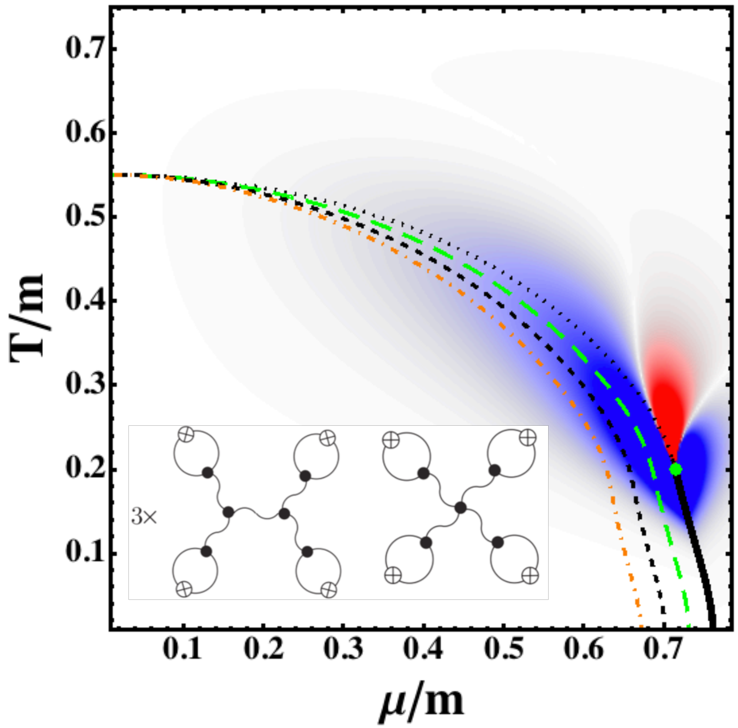
$$T^4 \frac{d^4}{d\mu^4} \log Z = -T^3 a_4^0 + 3T^2 (a_2^1)^2 < \delta\sigma^2 > + 4T^2 a_1^1 a_3^1 < \delta\sigma^2 > - 6T (a_1^1)^2 a_2^1 < \delta\sigma^3 >$$
$$+ (a_1^1)^4 (< \delta\sigma^4 > - 3 < \delta\sigma^2 >) - 12T \left[(a_1^1)^2 a_2^2 + 2 a_1^1 a_1^2 a_2^1 \right] < \delta\sigma^2 >^2$$
$$24 \left[(a_1^1)^3 a_1^3 + 2 (a_1^1)^2 (a_1^2)^2 \right] < \delta\sigma^2 >^3 + 24 (a_1^1)^3 a_1^2 < \delta\sigma^2 >< \delta\sigma^3 >$$





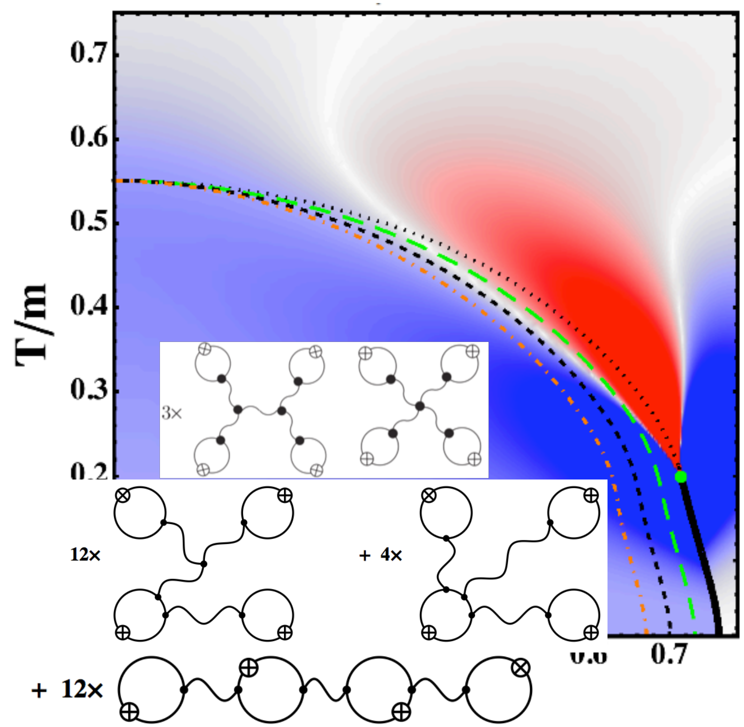
Ratio of the number susceptibilities

$$\kappa_4 = \langle \sigma_V^4 \rangle_c = 6VT^3[2(\lambda_3 \xi)^2 - \lambda_4]\xi^8$$



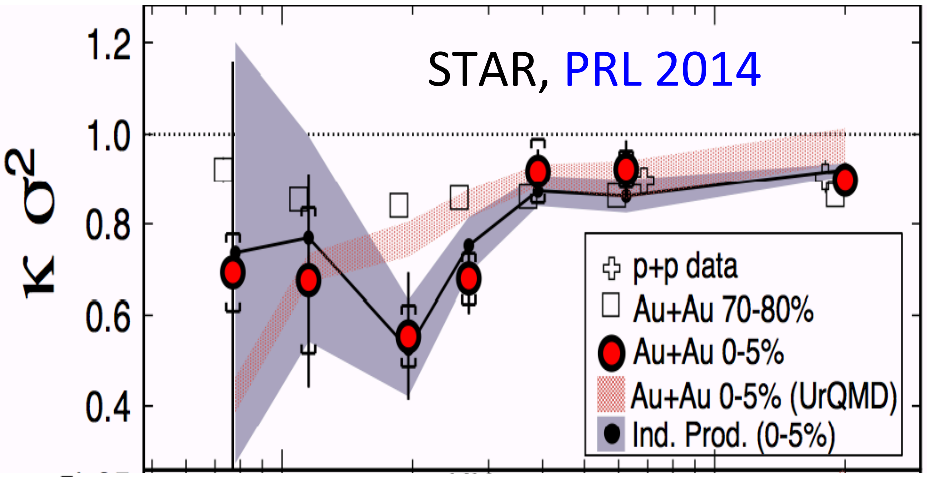
Kurtosis from order parameter fluctuation

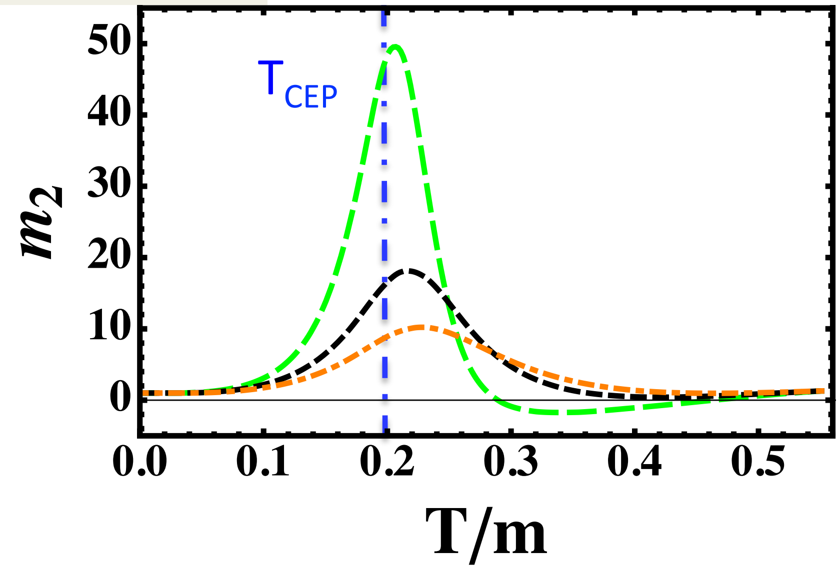
$$m_2 = \frac{T^4 \frac{d^4}{d\mu^4} \log Z}{T^2 \frac{d^2}{d\mu^2} \log Z}$$

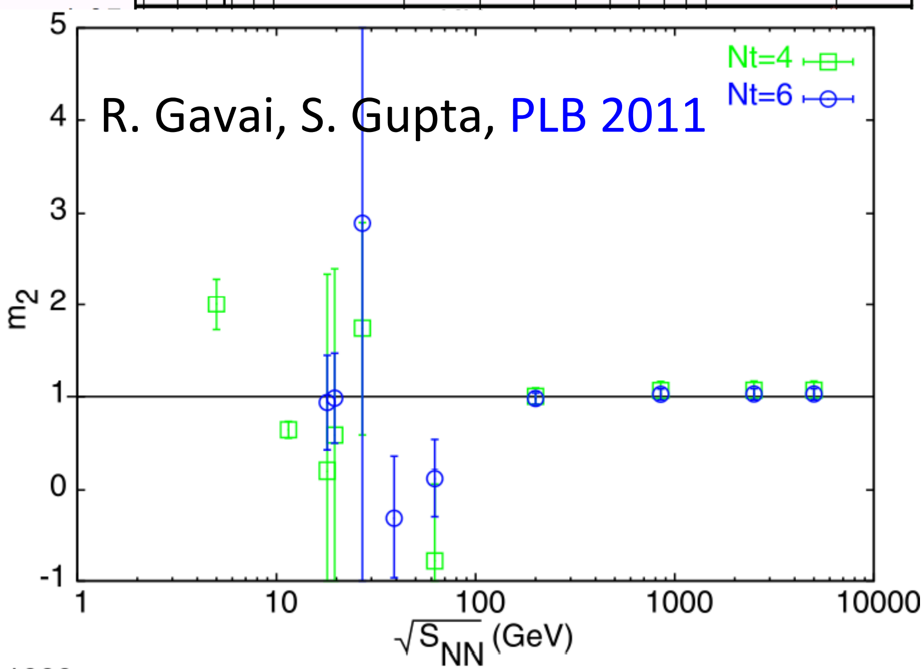


There is a large region of negative m_2, beginning at the critical point and opening up into the crossover region. The negative m_2 region overlaps with the "hadronic" phase near the critical point ➔ non-monotonic feature, sign change!

Comparison with m_2





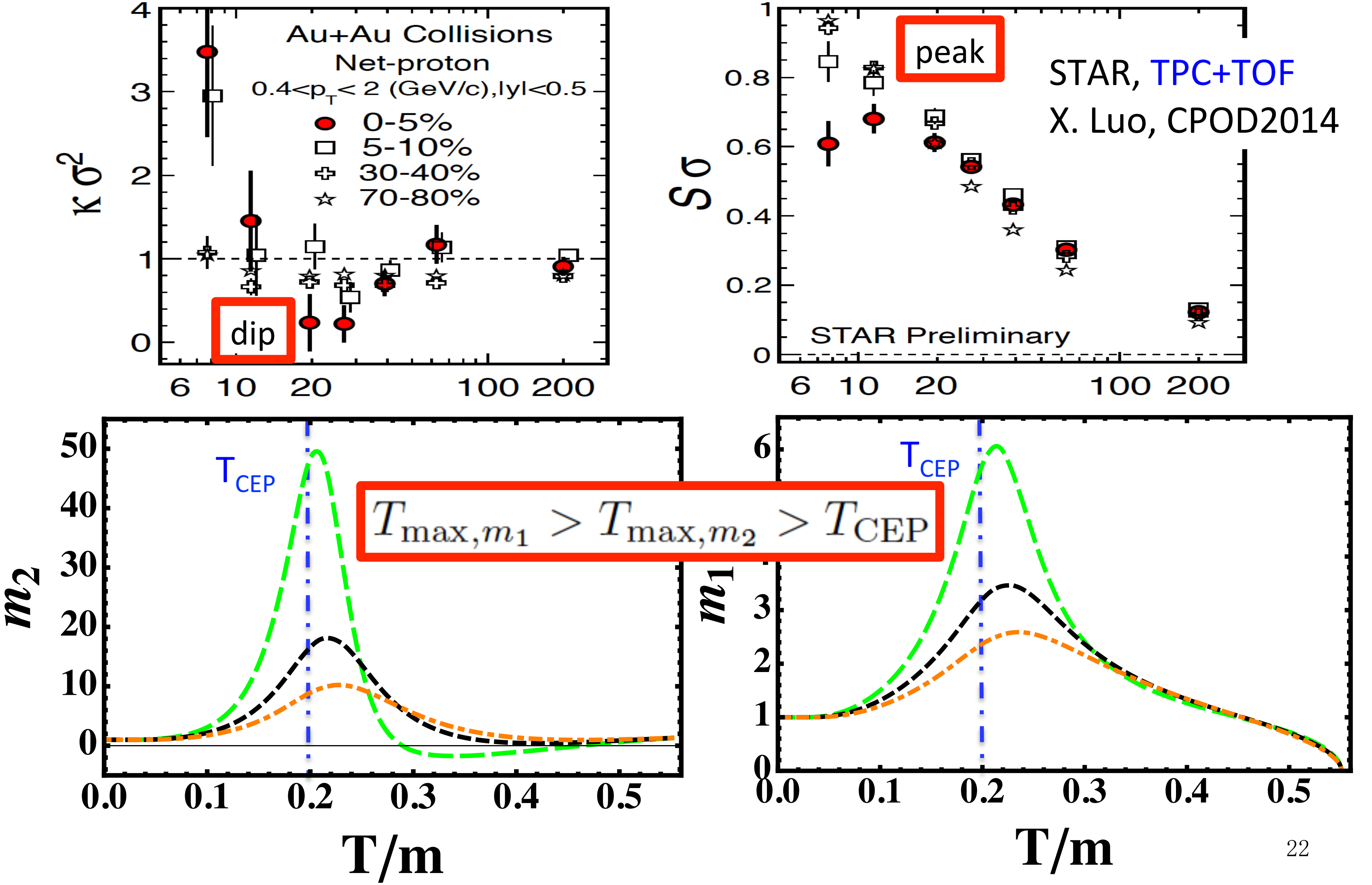


Our model calculation are qualitatively agree with the lattice and HIC data, our results are reasonable.

Near the CEP, m_2 has a large peak and may change its sign. The shape and magnitude depend on how close the freeze out line to the CEP.

More information/other probes are needed to localize the CEP, how about m_1?

Comparing with STAR new data



Further study with 3f-NJL model

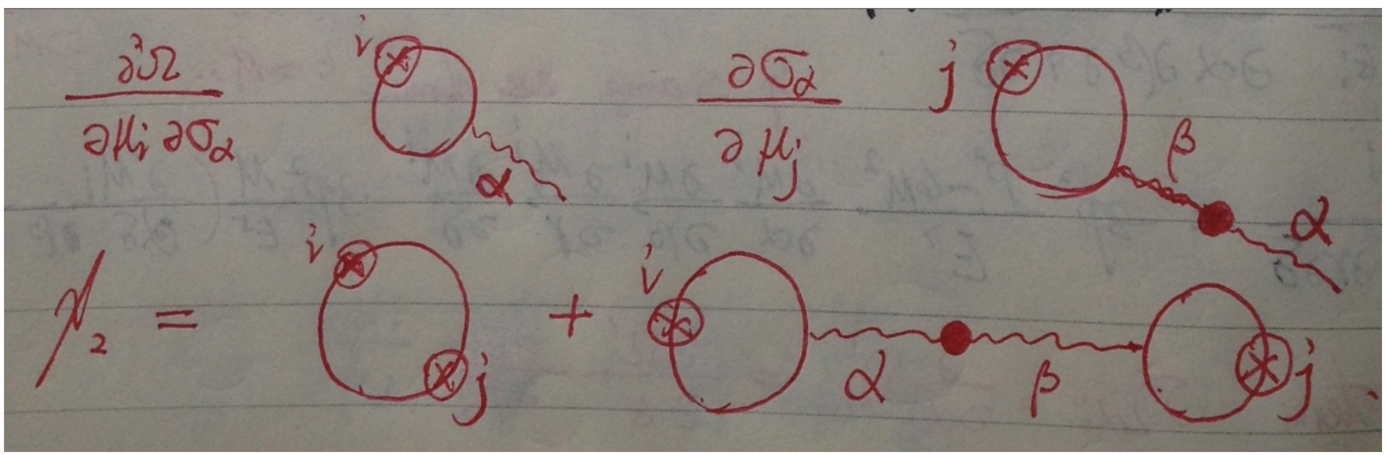
Effective potential:

$$\Omega(T,\mu_u,\mu_d,\mu_s) = 2G\left(\sigma_u^2+\sigma_d^2+\sigma_s^2\right) - 4K\sigma_u\sigma_d\sigma_s + \sum_{f=u,d,s}\Omega_f(T,\mu_f;m_f),$$

$$\Omega_f(T,\mu_f;m_f) = -2N_c\int\frac{d^3p}{(2\pi)^3}\left[E_f\Theta(\Lambda^2-\vec{p}^{\,2}) + T\ln\left[1+e^{-(E_f-\mu_f)/T}\right] + T\ln\left[1+e^{-(E_f+\mu_f)/T}\right]\right].$$

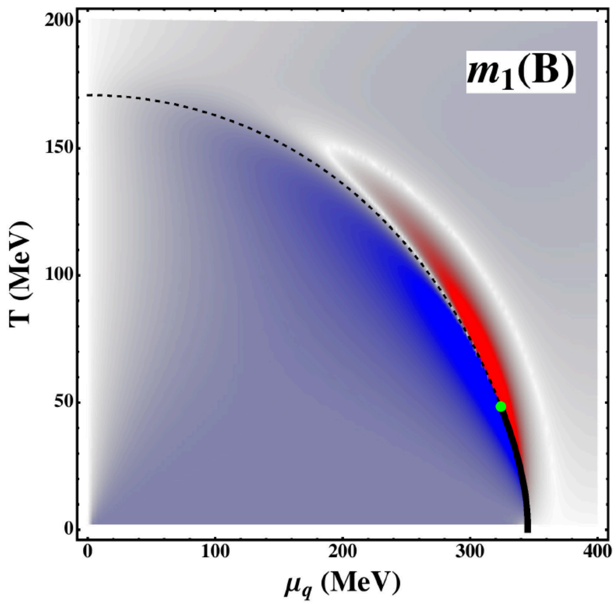
Flavor coupled:

$$E_f = \sqrt{m_f^2+p^2},\quad m_f = m_f^0 - 4G\sigma_f + 2K\sigma_{f'}\sigma_{f''},\qquad f\neq f'\neq f''\in\{u,d,s\}$$



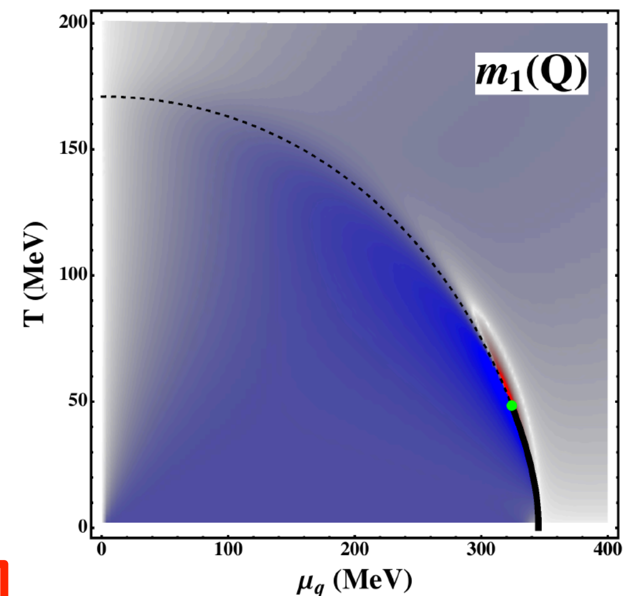
$$\chi_2^{ij} = \left.\frac{\partial^2\Omega}{\partial\mu_i\partial\mu_j}\right|_{\vec{\sigma}_0} - \sum_\alpha \left.\frac{\partial^2\Omega}{\partial\mu_i\partial\sigma_\alpha}\right|_{\vec{\sigma}_0}\left[\left.\frac{\partial^2\Omega}{\partial\sigma_\beta\partial\sigma_\alpha}\right|_{\vec{\sigma}_0}\right]^{-1}\left.\frac{\partial^2\Omega}{\partial\sigma_\beta\partial\mu_j}\right|_{\vec{\sigma}_0}$$

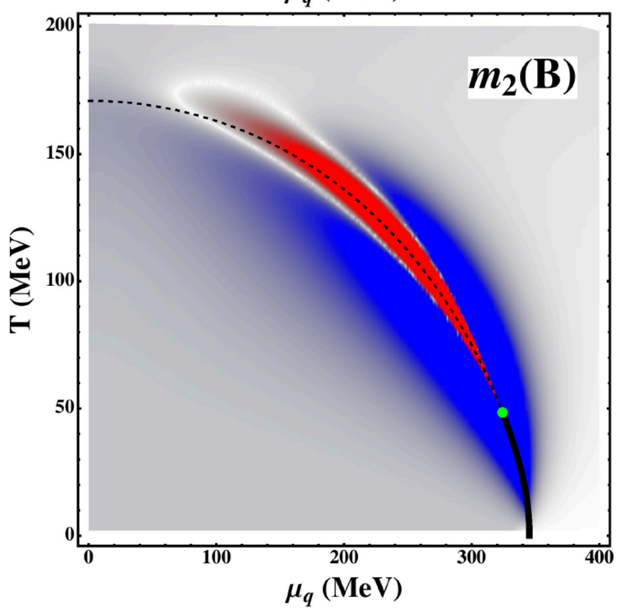
Susceptibilities with 3f-NJL model



Net Baryon

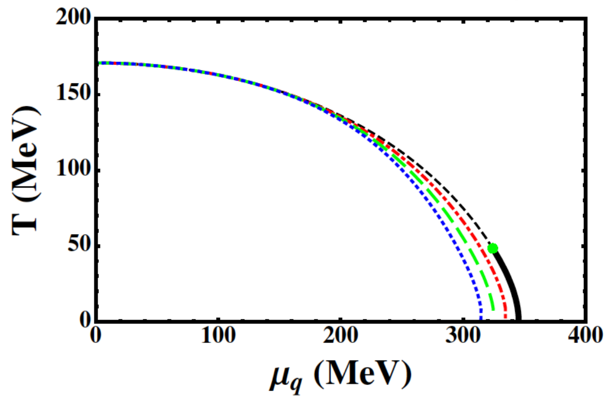
$$N_B = \frac{1}{3}(N_u + N_d + N_s)$$

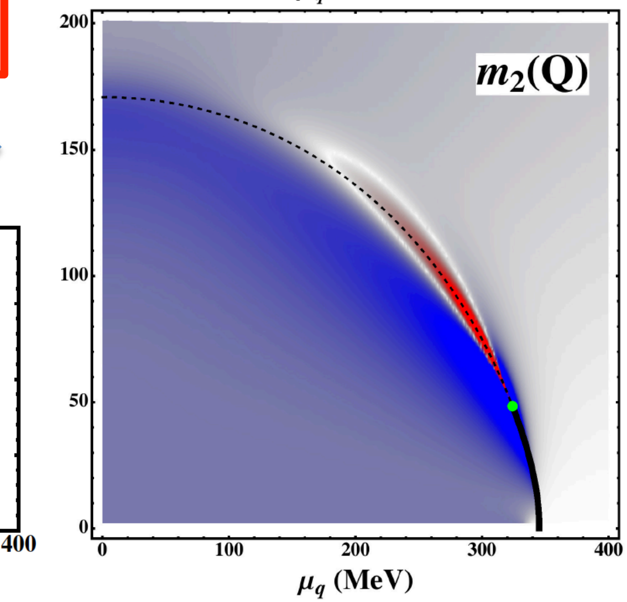




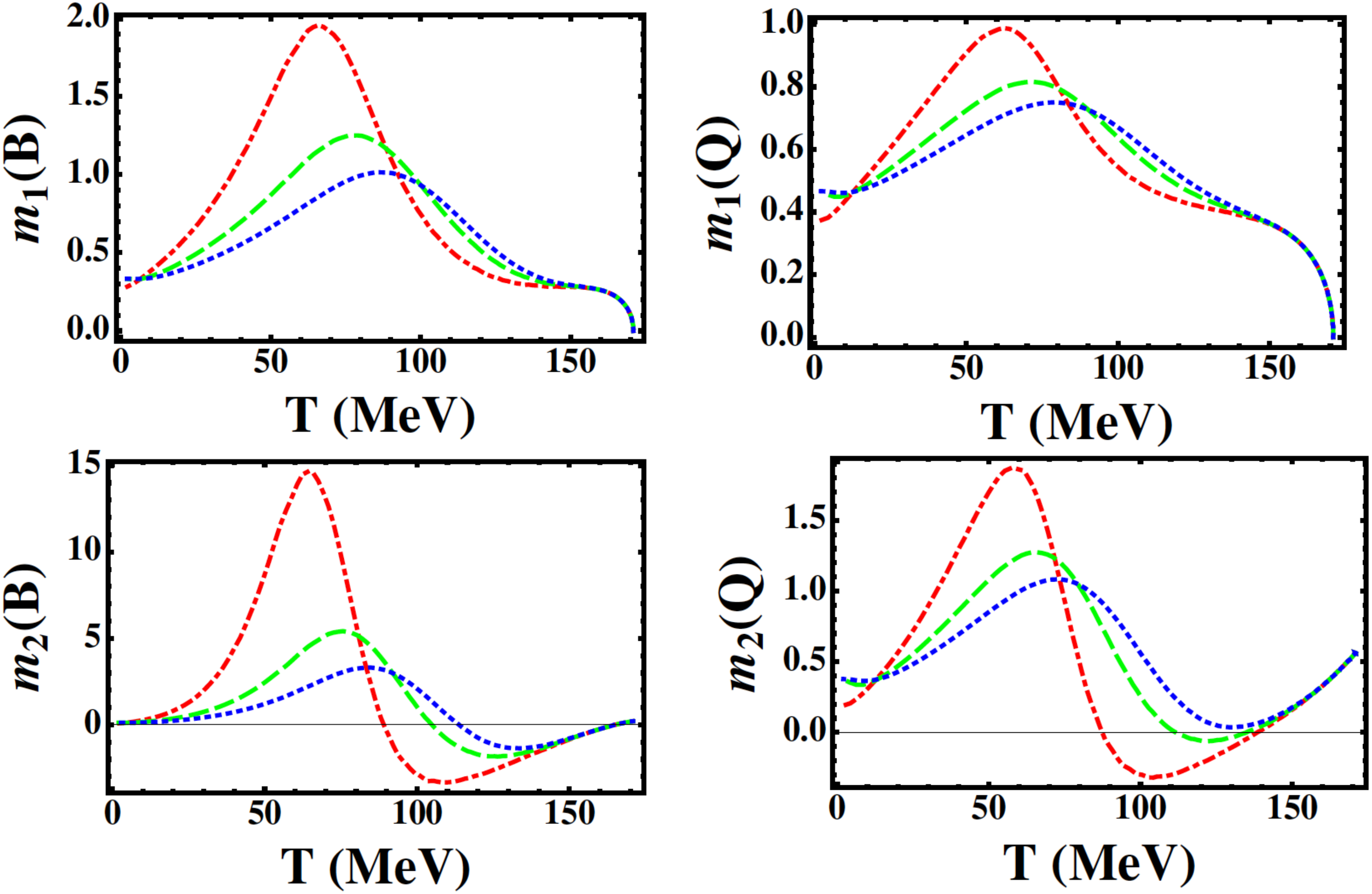
Net Charge

$$N_Q = \frac{1}{3}(2N_u - N_d - N_s)$$

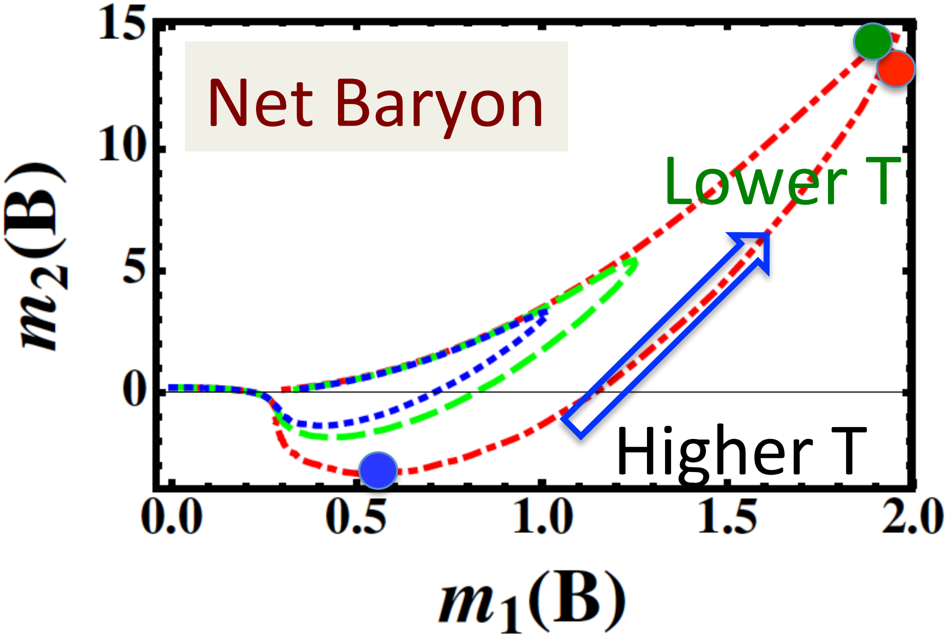


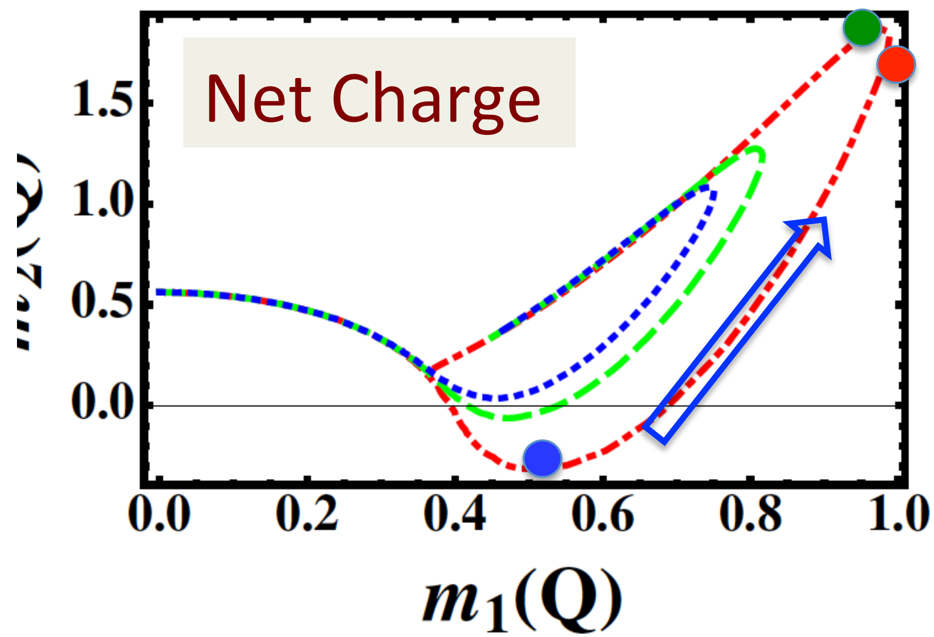


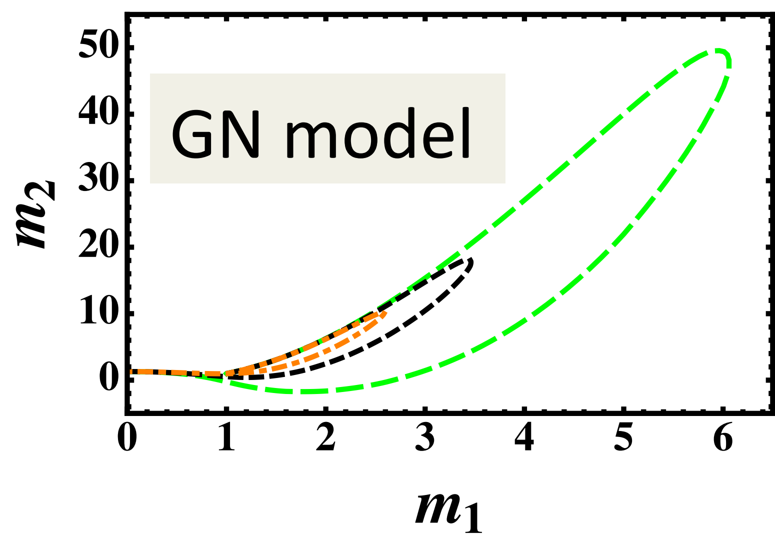
Along hypothetical Freeze-out lines



Combination of m_1 and m_2, common?



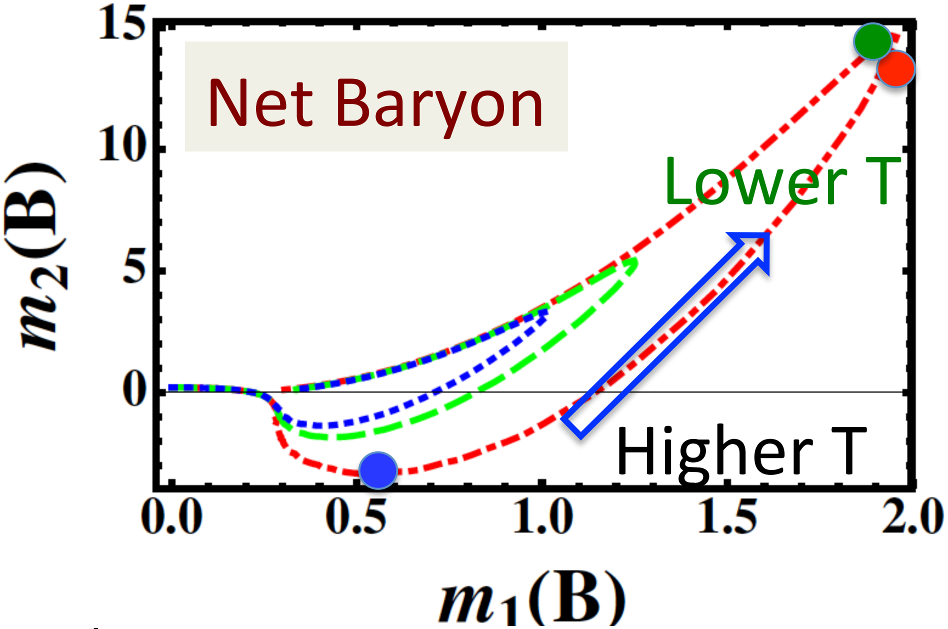


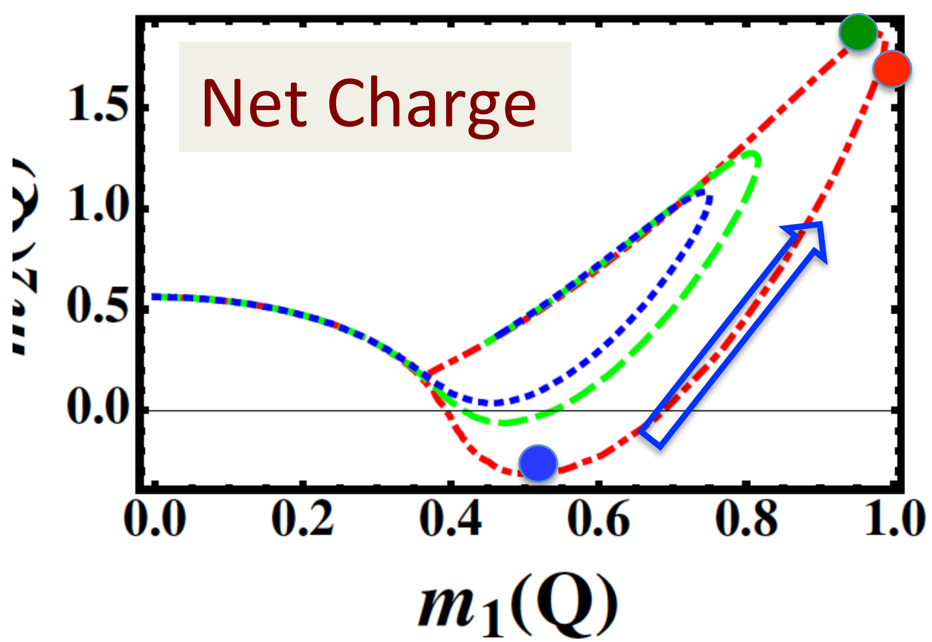


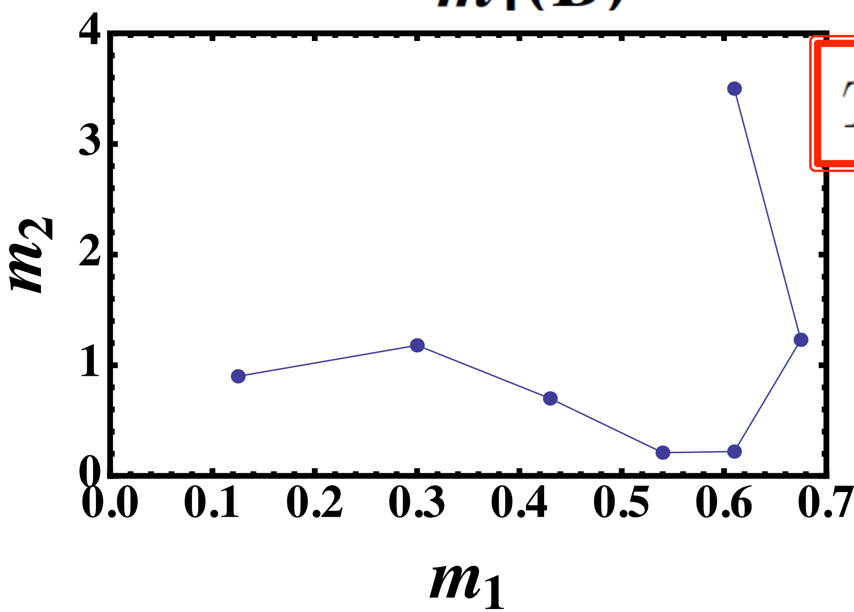
$$T_{\min,m_2} > T_{\max,m_1} > T_{\max,m_2} > T_{\mathrm{CEP}}$$

- also seen with Ising model
- robust features of CEP
- survive after non-thermal effects
- not seen for strangeness …

Combination of m_1 and m_2, common?







$$T_{\min,m_2} > T_{\max,m_1} > T_{\max,m_2} > T_{\text{CEP}}$$

- also seen with Ising model
- robust features of CEP
- survive after non-thermal effects
- not seen for strangeness …

Singularity with Strangeness

No net strange quark

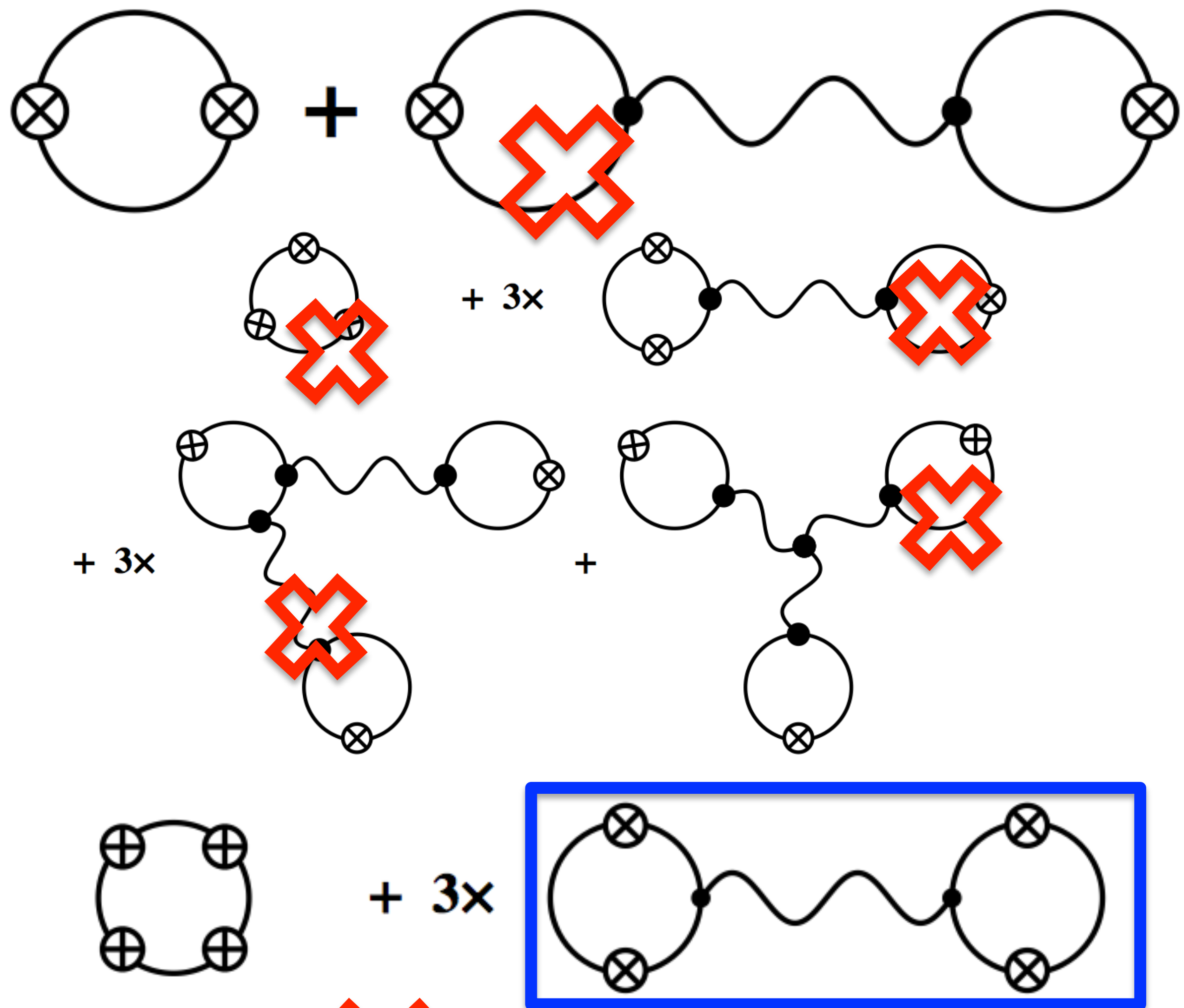
$$\mu_s = 0$$

Thermal suppression

$$\exp\left(-2\frac{m_s}{T}\right)$$

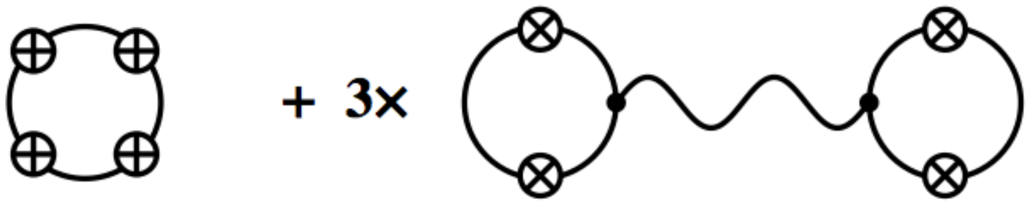
$$m_s \gg T$$

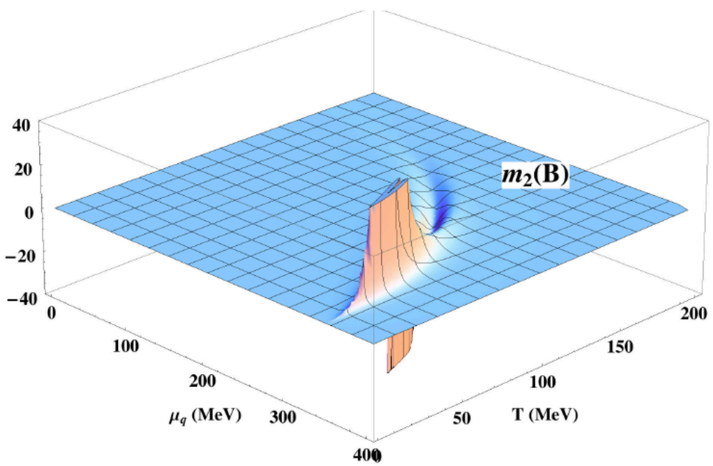
Singularity with strangeness is NOT visible except right on the CEP.

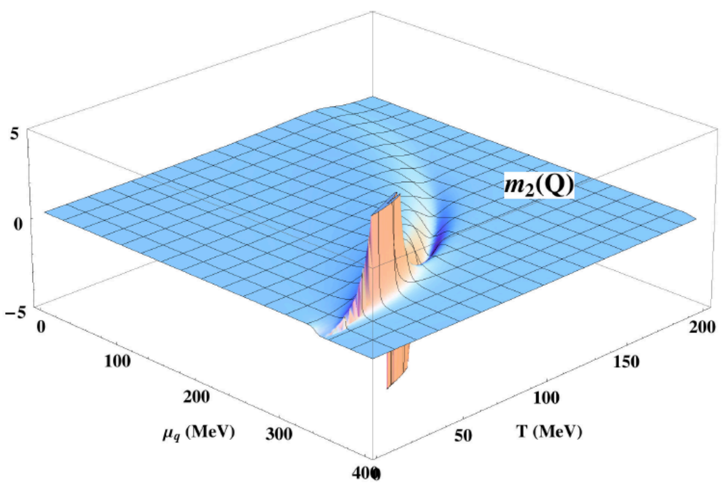


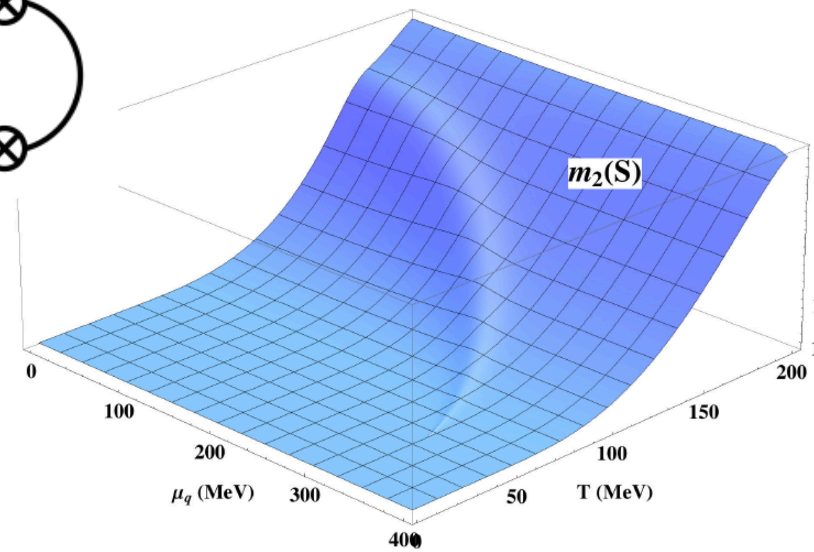
+ [60] other diagrams

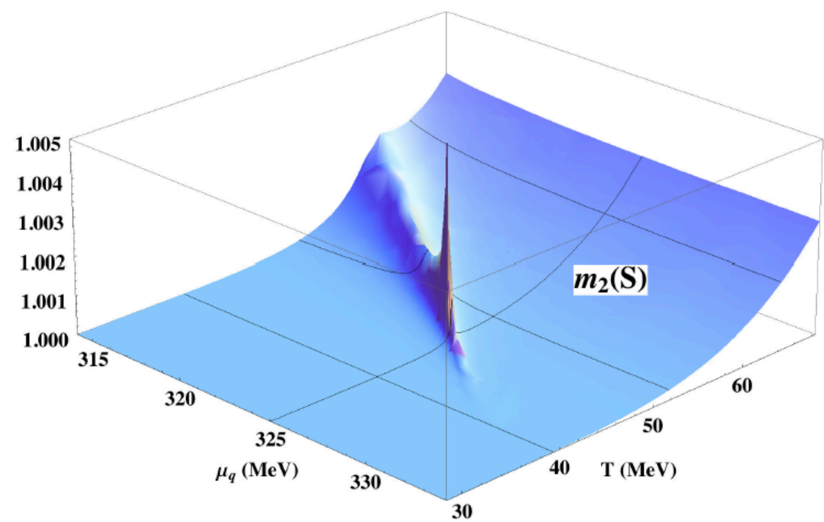
Suppressed singularity



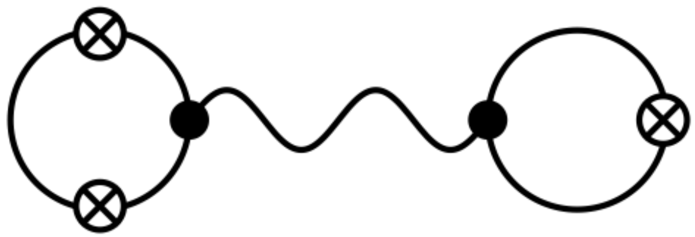


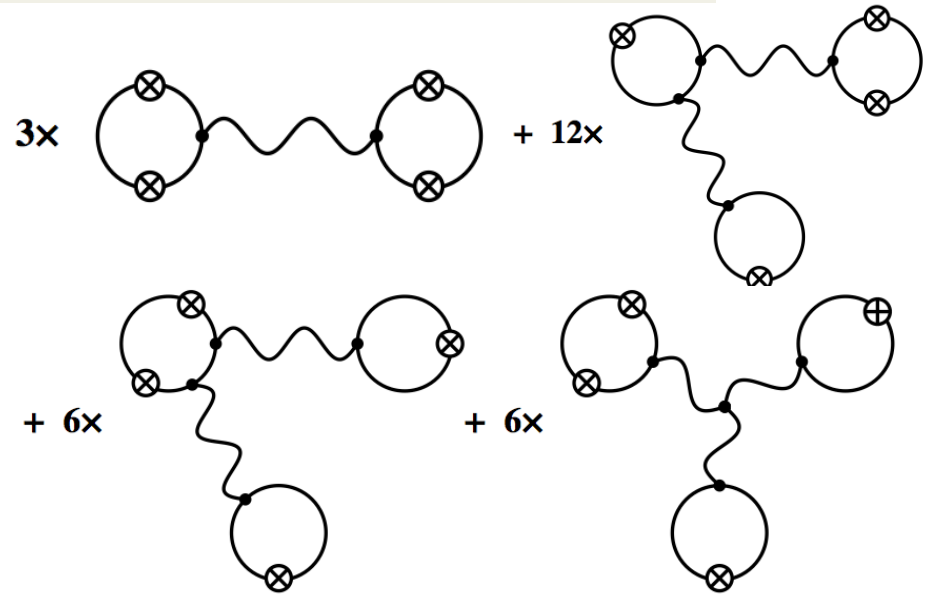


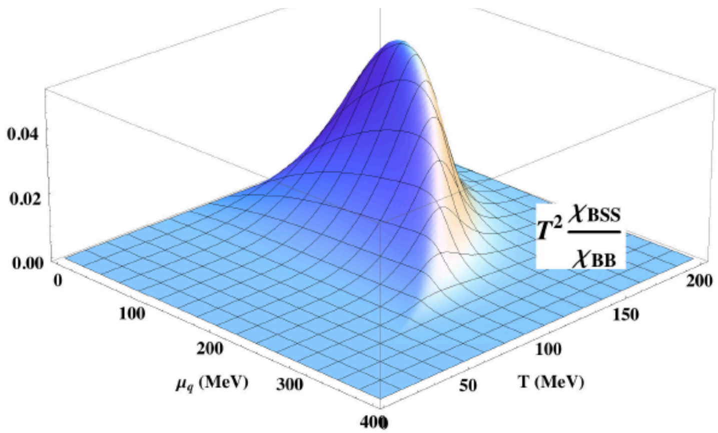


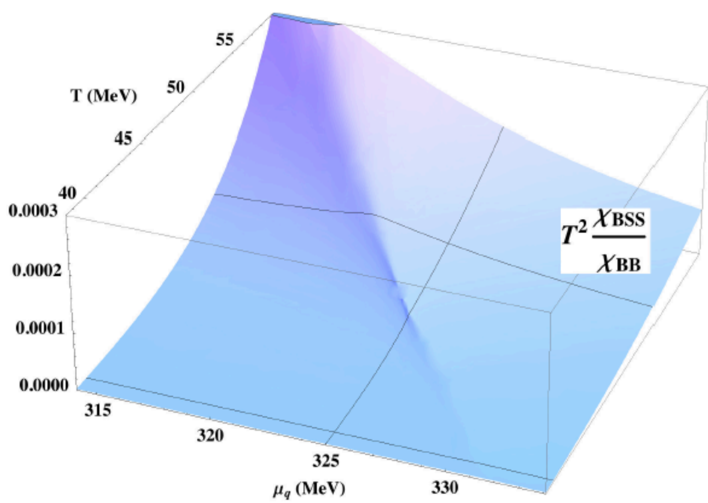


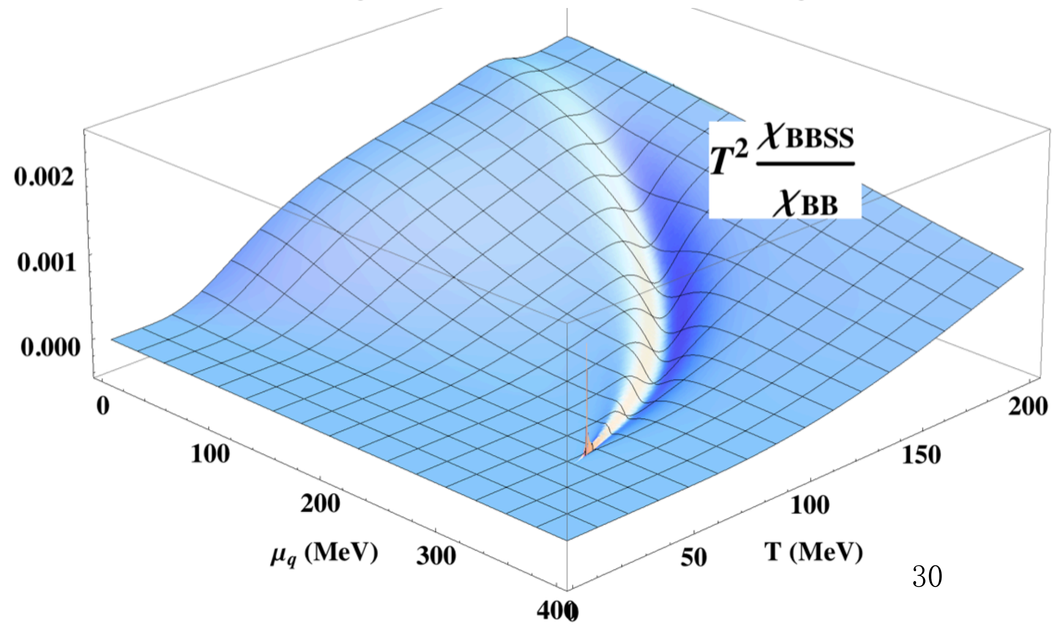
For mixing channels











Summary

1. Phase diagram with TCP and CEP are explored. With correlation length or discontinuity, singularity near CEP is universal.
2. Higher moments, susceptibilities and observables are discussed. Prediction with full tree-level correlators.
3. A large region of negative m_2, overlaps with the hadronic phase near CEP. Sign change of m_2, and peak in m_1 indicate non-monotonic behaviors. Flavor structure and possible observables(B, Q, S) are discussed.
4. Comparing with lattice and HIC data, non-monotonic features are likely to be robust. The shape of m_2 vs. m_1, and ordering $T_{min,m2} > T_{max,m1} > T_{max,m2} > T_{CEP}$ help to indicate the location of CEP.

Summary

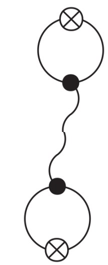
1. Phase diagram with TCP and CEP are explored. With correlation length or discontinuity, singularity near CEP is universal.
2. Higher moments, susceptibilities and observables are discussed. Prediction with full tree-level correlators.
3. A large region of negative m_2, overlaps with the hadronic phase near CEP. Sign change of m_2, and peak in m_1 indicate non-monotonic behaviors. Flavor structure and possible observables(B, Q, S) are discussed.
4. Comparing with lattice and HIC data, non-monotonic features are likely to be robust. The shape of m_2 vs. m_1, and ordering $T_{min,m2}>T_{max,m1}>T_{max,m2}>T_{CEP}$ help to indicate the location of CEP.

Thanks

Back up

Another way to check, Example with 2nd

$$T^2 \frac{d^2}{d\mu^2} \log Z \;=\; -T a_2^0 + {\color{red}(a_1^1)^2 < \delta\sigma^2 >} \qquad a_k^n = \frac{V}{n!} \frac{\partial^{k+n}\Omega}{\partial\mu^k \partial\sigma^n}$$



With $P(T,\mu) = -\Omega(T,\mu,\sigma_0) = -\frac{T}{V}\log Z$

$$\frac{\mathbf{d}\Omega(\mu,T,\sigma_0(\mu,T))}{\mathbf{d}\mu} \equiv \lim_{\Delta\mu\to 0} \frac{\Omega(\mu+\Delta\mu,T,\sigma_0(\mu+\Delta\mu,T)) - \Omega(\mu,T,\sigma_0(\mu,T))}{\Delta\mu}$$

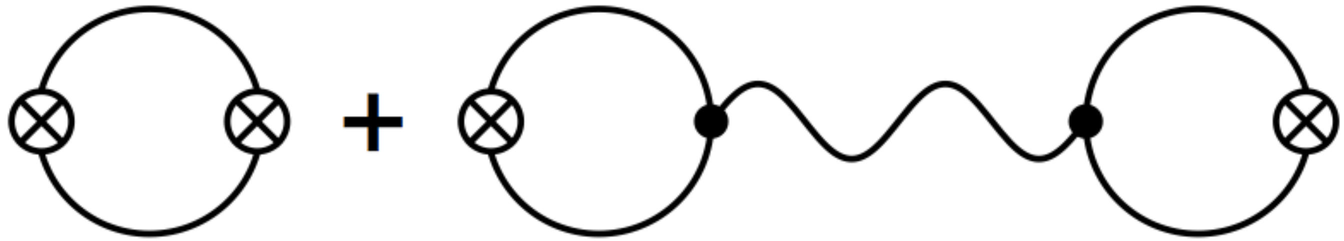
with gap equation: $\frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\mu\partial\sigma} + \frac{\partial^2\Omega(\mu,T,\sigma_0)}{\partial\sigma^2}\frac{\partial\sigma_0}{\partial\mu} = 0$

$$\frac{\mathbf{d^2}\Omega(\mu,T,\sigma_0(\mu,T))}{\mathbf{d}\mu^2} = \left.\frac{\partial^2\Omega}{\partial\mu^2}\right|_{\sigma=\sigma_0(\mu,T)} + \left.\frac{\partial^2\Omega}{\partial\sigma\partial\mu}\right|_{\sigma=\sigma_0(\mu,T)} \frac{\partial\sigma_0(\mu,T)}{\partial\mu}$$

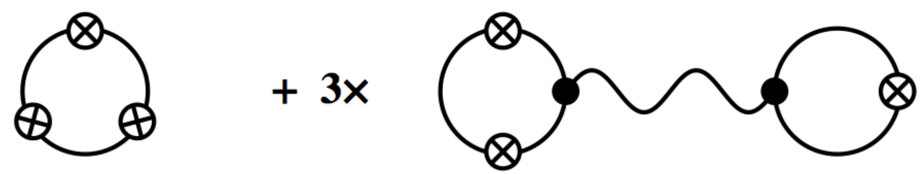
$$= \frac{1}{V}a_2^0 - \frac{1}{V^2}\frac{(a_1^1)^2}{m_\sigma^2} = \frac{1}{V}a_2^0 - \frac{1}{VT}(a_1^1)^2\langle\sigma_0^2\rangle$$

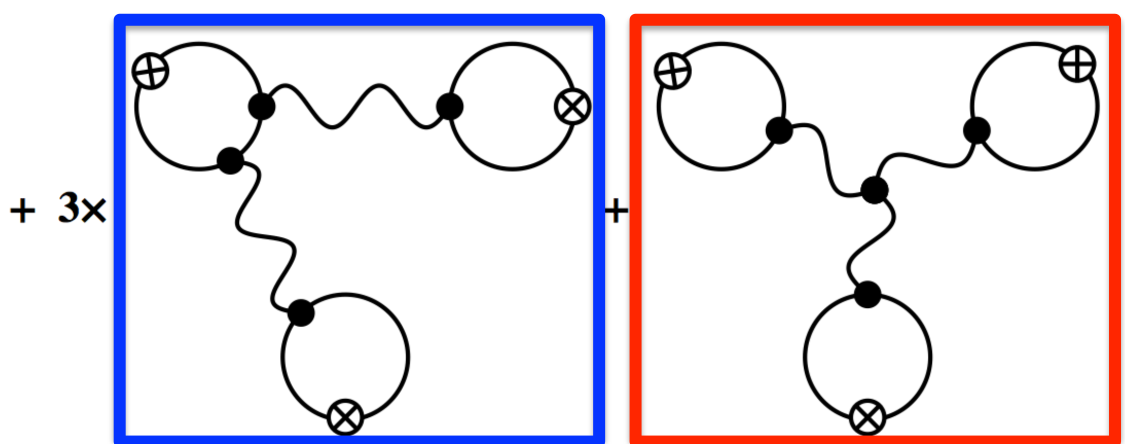
Tree level contribution

$$T^2 \frac{d^2}{d\mu^2} \log Z \;=\; -T a_2^0 + (a_1^1)^2 < \delta\sigma^2 >$$



$$T^3 \frac{d^3}{d\mu^3} \log Z \;=\; -T^2 a_3^0 + 3T a_1^1 a_2^1 < \delta\sigma^2 > - (a_1^1)^3 < \delta\sigma^3 > - 6 (a_1^1)^2 a_1^2 < \delta\sigma^2 >^2$$





$$
\begin{aligned}
T^4 \frac{d^4}{d\mu^4} \log Z \;=\; & -T^3 a_4^0 + 3T^2 (a_2^1)^2 < \delta\sigma^2 > + 4T^2 a_1^1 a_3^1 < \delta\sigma^2 > - 6T (a_1^1)^2 a_2^1 < \delta\sigma^3 > \\
& + (a_1^1)^4 \left(< \delta\sigma^4 > - 3 < \delta\sigma^2 > \right) - 12T \left[(a_1^1)^2 a_2^2 + 2 a_1^1 a_1^2 a_2^1 \right] < \delta\sigma^2 >^2 \\
& + 24 \left[(a_1^1)^3 a_1^3 + 2 (a_1^1)^2 (a_1^2)^2 \right] < \delta\sigma^2 >^3 + 24 (a_1^1)^3 a_1^2 < \delta\sigma^2 >< \delta\sigma^3 >
\end{aligned}
$$